

COVARIANTS, INVARIANT SUBSETS, AND FIRST INTEGRALS

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ABSTRACT. Let \mathbb{k} be an algebraically closed field of characteristic 0, and let V be a finite-dimensional vector space. Let $\text{End}(V)$ be the semigroup of all polynomial endomorphisms of V . Let $\mathcal{E} \subseteq \text{End}(V)$ be a linear subspace which is also a semi-subgroup. Both $\text{End}(V)$ and \mathcal{E} are ind-varieties which act on V in the obvious way.

In this paper, we study important aspects of such actions. We assign to \mathcal{E} a linear subspace $\mathcal{D}_{\mathcal{E}}$ of the vector fields on V . A subvariety X of V is said to be $\mathcal{D}_{\mathcal{E}}$ -invariant if $\xi(x) \in T_x X$ for all $\xi \in \mathcal{D}_{\mathcal{E}}$. We show that X is $\mathcal{D}_{\mathcal{E}}$ -invariant if and only if it is the union of \mathcal{E} -orbits. For such X , we define first integrals and construct a quotient space for the \mathcal{E} -action.

An important case occurs when G is an algebraic subgroup of $\text{GL}(V)$ and \mathcal{E} consists of the G -equivariant polynomial endomorphisms. In this case, the associated $\mathcal{D}_{\mathcal{E}}$ is the space the G -invariant vector fields. A significant question here is whether there are non-constant G -invariant first integrals on X . As examples, we study the adjoint representation, orbit closures of highest weight vectors, and representations of the additive group. We also look at finite-dimensional irreducible representations of SL_2 and its nullcone.

1. INTRODUCTION AND MAIN RESULTS

The study of algebraic group actions, which began in the nineteenth century with the action of SL_2 on binary forms, has proved to be incredibly fertile. In this paper, we consider some important aspects of that subject in a new setting, ind-varieties (see Appendix). Let \mathbb{k} be an algebraically closed field of characteristic 0, and let X be an irreducible affine variety over \mathbb{k} . Let \mathcal{D} be a set of vector fields on X . A closed subvariety $Y \subseteq X$ is called \mathcal{D} -invariant if $\xi(y) \in T_y Y$ for all $y \in Y$ and $\xi \in \mathcal{D}$. We establish some basic properties of invariant subsets including the following: for any $x \in X$, there is a smallest \mathcal{D} -invariant closed subvariety, $M(x)$, which contains x (Lemma 2.5). When \mathcal{D} is a linear subspace, we define a first integral of \mathcal{D} to be a function $f \in \mathbb{k}(X)$ such that $\xi f = 0$ for all $\xi \in \mathcal{D}$. We show that first integrals are precisely those functions which are constant on the spaces $M(x)$ (Lemma 5.2).

We next consider the semigroup $\text{End}(X)$ consisting of all endomorphisms of X . An important fact is that $\text{End}(X)$ is an ind-variety. This allows us to define the (Zariski) tangent space $T_{\text{id}} \text{End}(X)$ and to associate to any $A \in T_{\text{id}} \text{End}(X)$ a vector field ξ_A on X (section 3.1). For $\mathcal{E} \subseteq \text{End}(X)$, a closed semi-subgroup, we denote by $\mathcal{D}_{\mathcal{E}}$ the associated vector fields. The \mathcal{E} -orbit of an element $x \in X$ is defined as $\mathcal{E}(x) := \{\varphi(x) \mid \varphi \in \mathcal{E}\}$. We first show that if a closed subvariety $Y \subseteq X$ is the union of \mathcal{E} -orbits, then Y is $\mathcal{D}_{\mathcal{E}}$ -invariant (Proposition 3.2).

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Our main application of these ideas is to the case where X is a finite dimensional vector space and Y is a closed subvariety. We change notation and denote the vector space by V and the subvariety by X . We assume from now on that $\mathcal{E} \subseteq \text{End}(V)$ is a linear subspace which is also a semi-subgroup. In this context, we show that for $v \in V$, $\mathcal{E}(v) = M(v)$ and that a subvariety $X \subseteq V$ is $\mathcal{D}_{\mathcal{E}}$ -invariant if and only if it is the union of \mathcal{E} -orbits (Theorem 3.7). This means that $\mathcal{D}_{\mathcal{E}}$ -invariant subvarieties are precisely those which are stable under the action of \mathcal{E} . Furthermore, first integrals separate \mathcal{E} -orbits on an open set and can be used to construct a quotient space for the action of \mathcal{E} on X (Proposition 5.5). This construction includes an algebraic (and global) version of a classical theorem of FROBENIUS [War71, Theorem 1.60].

The most important instances of the above occur when G is an algebraic group acting linearly on V , and \mathcal{E} is the semigroup of covariants, i.e.,

$$\mathcal{E} = \text{End}_G(V) := \{\varphi \in \text{End}(V) \mid \varphi(g \cdot v) = g \cdot \varphi(v) \text{ for all } g \in G \text{ and } v \in V\}.$$

An important question here is whether or not there are non-constant G -invariant first integrals. Examples show that such can occur. However, in those cases where they do not, the field of first integrals is the field of rational functions on a homogeneous space (Lemma 5.8 and Theorem 5.16).

When G is reductive and the orbit Gv is closed, $\mathcal{E}(v)$ can be described in terms of the stabilizer of v (Proposition 4.10). Furthermore, when the generic G -orbit in X is closed and $\mathcal{E} = \text{End}_G(V)$, we show that there are no G -invariant first integrals (Theorem 5.16). Finally, in section 6, we study the case where $G = \text{SL}_2$ and either $X = V_d$, the binary forms of degree d , or X is the nullcone of V_d .

The background for some of the ideas in this paper comes from ordinary differential equations, see [GSW12]. In that context, it can be shown that a Zariski-closed set X is $\mathcal{D}_{\mathcal{E}}$ -invariant if and only if it is the union of trajectories of solutions to $\frac{dx}{dt} = F(x)$, $F \in \mathcal{E}$. The quotient construction given in this paper leads to a description of all such sets. The study of the $\mathcal{E}(v)$ began with the paper [LS99] by LEHRER and SPRINGER which was subsequently extended by PANYUSHEV in [Pan02]. The difficult problem of constructing the module of covariants on X was first considered in the nineteenth century [Ell13] and continues to be of interest [Dom08]. In this paper, we consider that problem in the context of finding the maximum dimension of the orbits $\mathcal{E}(v)$.

2. BASIC MATERIAL

2.1. Vector fields and \mathcal{D} -invariant subsets. Our base field \mathbb{k} is algebraically closed of characteristic zero. We start with a lemma which translates the concept of invariant subsets with respect to an ordinary differential equation into the algebraic setting. For an affine variety X an *algebraic vector field* $\xi = (\xi(x))_{x \in X}$ is a collection of tangent vectors $\xi(x) \in T_x X$ such that, for every regular function $f \in \mathcal{O}(X)$, the function $\xi f: x \mapsto \xi(x)f$ is again regular. It is easy to see that this is the same as a derivation of the coordinate ring $\mathcal{O}(X)$. Note that ξf is also defined for rational functions f .

In addition, one can define the *tangent bundle* TX of X which is a variety together with a projection $p: TX \rightarrow X$ such that the fibers $p^{-1}(x)$ are the Zariski tangent spaces $T_x X$. Then the sections are the algebraic vector fields (see e.g.

[Kra14, Appendix A.4.5]). It is clear that the algebraic vector fields form a $\mathcal{O}(X)$ -module which will be denoted by $\text{Vec}(X)$ and which can be identified with the $\mathcal{O}(X)$ -module $\text{Der}(\mathcal{O}(X))$ of derivations of $\mathcal{O}(X)$.

Lemma 2.1. *Let X be a smooth complex variety, and let $\xi \in \text{Vec}(X)$ be an algebraic vector field. Then a Zariski-closed subvariety $Y \subseteq X$ is invariant with respect to the flow defined by the differential equation $\dot{x} = \xi(x)$ if and only if $\xi(y) \in T_y Y$ for all $y \in Y$.*

Proof. Let $\Phi: X \times \mathbb{R} \rightarrow X$ be the local flow of ξ , defined in an open neighborhood of $X \times \{0\}$. By definition,

$$\frac{\partial}{\partial t} \Phi(x, t)|_{t=0} = \xi(x) \text{ for all } x \in X.$$

This implies that if Y is invariant under Φ , then $\xi(y) \in T_y Y$ for all $y \in Y$. On the other hand, assume that $\xi(y) \in T_y Y$ for all $y \in Y$, and denote by $Y' \subseteq Y$ the open dense set of smooth points of Y . Then $\xi|_{Y'}$ defines a local flow $\Phi_{Y'}: Y' \times \mathbb{R} \rightarrow Y'$ such that $\frac{\partial}{\partial t} \Phi_{Y'}(y', t)|_{t=0} = \xi(y')$ for all $y' \in Y'$. By the uniqueness of the local flow, we have $\Phi_{Y'} = \Phi|_{Y' \times \mathbb{R}}$, and so Y' is invariant under Φ . Since $Y = \overline{Y'}$ we see that Y is also invariant under Φ . \square

This lemma allows to define the invariance of subvarieties with respect to a set of vector fields for an arbitrary \mathbb{k} -variety X .

Definition 2.2. Let $\mathcal{D} \subseteq \text{Vec}(X)$ be a set of vector fields.

- (1) A closed subvariety $Y \subseteq X$ is called *\mathcal{D} -invariant* if $\xi(y) \in T_y Y$ for all $y \in Y$ and all $\xi \in \mathcal{D}$. We also say that the vector fields $\xi \in \mathcal{D}$ are *parallel to Y* .
- (2) A subspace $W \subseteq \mathcal{O}(X)$ is called *\mathcal{D} -invariant* if $\xi(W) \subseteq W$ for all $\xi \in \mathcal{D}$.

Remark 2.3. We will constantly use the following easy fact. If ξ is a vector field parallel to Y and f a rational function on X defined in a neighborhood U of $y \in Y$, then $\xi(y)f = \xi(y)(f|_{U \cap Y})$. In particular, if f is regular on X , then $(\xi f)|_Y = \xi|_Y(f|_Y)$.

2.2. \mathcal{D} -invariant ideals. Let $\mathcal{D} \subseteq \text{Vec}(X)$ be a set of vector fields.

Lemma 2.4. *If $I(Y) \subseteq \mathcal{O}(X)$ denotes the vanishing ideal of Y , then Y is \mathcal{D} -invariant if and only if $I(Y)$ is \mathcal{D} -invariant.*

Proof. If $f \in I(Y)$, then, for $y \in Y$, $(\xi f)(y) = \xi(y)f = \xi(y)f|_Y = 0$, hence $\xi f \in I(Y)$. Conversely, if $\xi(I(Y)) \subseteq I(Y)$, then ξ induces a derivation of $\mathcal{O}(X)/I(Y) = \mathcal{O}(Y)$, and the claim follows. \square

For a closed subvariety $Y \subseteq X$ we can define the Lie subalgebra of the vector fields on X parallel to Y :

$$\text{Vec}_Y(X) := \{\xi \in \text{Vec}(X) \mid \xi(y) \in T_y Y \text{ for all } y \in Y\} \subseteq \text{Vec}(X).$$

We have a surjective homomorphism of Lie algebras

$$\rho: \text{Vec}_Y(X) \rightarrow \text{Vec}(Y), \quad \xi \mapsto \xi|_Y,$$

whose kernel consists of the vector fields on X vanishing on Y . In order to see that this map is surjective it suffices to consider the case where X is a vector space, and then the statement is clear. With this notation we see that Y is \mathcal{D} -invariant if and only if $\mathcal{D} \subseteq \text{Vec}_Y(X)$.

- Lemma 2.5.** (1) *Sums and intersections of \mathcal{D} -invariant ideals are \mathcal{D} -invariant.*
 (2) *If $I \subseteq \mathcal{O}(X)$ is a \mathcal{D} -invariant ideal, then so is \sqrt{I} .*
 (3) *If $Y_i \subseteq X$, $i \in I$, are \mathcal{D} -invariant closed subvarieties, then so is $\bigcap_{i \in I} Y_i$.*
 (4) *For any $x \in X$ there is a uniquely defined minimal \mathcal{D} -invariant closed subvariety $M(x) \subseteq X$ containing x .*
 (5) *If the closed subvariety $Y \subseteq X$ is \mathcal{D} -invariant, then every irreducible component of Y is \mathcal{D} -invariant.*

Proof. (1) is clear, (3) follows from (1) and (2), and (4) follows from (3).

(2) It suffices to show that if $f^n = 0$, then $(\xi f)^m = 0$ for some $m > 0$. Let $e_0 \geq 0$ be the minimal e such that there exists a $q \geq 0$ with $f^e \cdot (\xi f)^q = 0$. If $e_0 = 0$, we are done. So assume that $e_0 > 0$. Then

$$0 = \xi(f^{e_0} \cdot (\xi f)^q) \cdot \xi f = e_0 f^{e_0-1} \cdot (\xi f)^{q+1} + q f^{e_0} \cdot (\xi f)^q \cdot \xi^2 f = e_0 f^{e_0-1} \cdot (\xi f)^{q+1},$$

contradicting the minimality of e_0 .

(5) It suffices to consider the case where $Y = X$, hence $(0) = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_k$ where the \mathfrak{p}_i are the minimal primes of $\mathcal{O}(X)$. For every i choose an element $p_i \in \bigcap_{j \neq i} \mathfrak{p}_j \setminus \mathfrak{p}_i$. Then $\mathfrak{p}_i = \{p \in \mathcal{O}(X) \mid p_i p = 0\}$, and the same holds for every power of p_i . For every $p \in \mathfrak{p}_i$ we find

$$0 = p_i \xi(p_i p) = p_i(p_i \xi p + p \xi p_i) = p_i^2 \xi p,$$

hence $\xi p \in \mathfrak{p}_i$. □

Definition 2.6. The closed subvarieties $M(x) \subseteq X$ from Lemma 2.5(4) are called *minimal \mathcal{D} -invariant subvarieties*. By Lemma 2.5(5) they are irreducible.

2.3. Linear spaces of vector fields. In the following, we will mainly deal with the case where $\mathcal{D} \subseteq \text{Vec}(X)$ is a linear subspace. In this case, we set

$$\mathcal{D}(x) := \varepsilon_x(\mathcal{D}) := \{\xi(x) \mid \xi \in \mathcal{D}\} \subseteq T_x X$$

where $\varepsilon_x: \text{Vec}(X) \rightarrow T_x X$ is the (linear) evaluation map $\xi \mapsto \xi(x)$. Note that a closed subvariety $Y \subseteq X$ is \mathcal{D} -invariant if and only if $\mathcal{D}(y) \subseteq T_y Y$ for all $y \in Y$.

The following lemma is clear.

Lemma 2.7. *For a linear subspace $\mathcal{D} \subseteq \text{Vec}(X)$ the function $x \mapsto \dim \mathcal{D}(x)$ is lower semicontinuous, i.e., for every $x \in X$ the set $U_x := \{u \in X \mid \dim \mathcal{D}(u) \geq \dim \mathcal{D}(x)\}$ is a (Zariski-) open neighborhood of x .*

Setting $d_{\mathcal{D}}(X) := \max_{x \in X} \dim \mathcal{D}(x)$ the lemma implies that

$$X' := \{x \in X \mid \dim \mathcal{D}(x) = d_{\mathcal{D}}(X)\}$$

is open (and non-empty) in X .

3. ENDOMORPHISMS

3.1. The semigroup of endomorphisms. We now study the semigroup $\text{End}(X)$ of endomorphisms of X . An important fact is that $\text{End}(X)$ is an *ind-variety* (see Appendix) which allows to define the (Zariski) tangent space $T_{\text{id}} \text{End}(X)$. We have a canonical inclusion

$$\Xi: T_{\text{id}} \text{End}(X) \hookrightarrow \text{Vec}(X), \quad A \mapsto \xi_A,$$

where the vector field ξ_A is defined in the following way (see Appendix, Proposition 7.18). For any $x \in X$ consider the “orbit map” $\mu_x: \text{End}(X) \rightarrow X$, $\varphi \mapsto \varphi(x)$, and its differential

$$(d\mu_x)_{\text{id}}: T_{\text{id}} \text{End}(X) \rightarrow T_x X.$$

Then define $\xi_A(x) := (d\mu_x)_{\text{id}}(A)$.

Note that $\text{End}(X)$ acts on X ,

$$\Phi: \text{End}(X) \times X \rightarrow X, \quad (\varphi, x) \mapsto \varphi(x),$$

and this action is a morphism of ind-varieties. For the differential we find

$$(*) \quad d\Phi_{(\text{id}, x_0)}: T_{\text{id}} \text{End}(X) \oplus T_{x_0} X \rightarrow T_{x_0} X, \quad (A, \delta) \mapsto \xi_A(x_0) + \delta.$$

Definition 3.1. If $\mathcal{E} \subseteq \text{End}(X)$ is a closed semi-subgroup we say that a subset $Y \subseteq X$ is *stable under \mathcal{E}* (shortly \mathcal{E} -stable), if $\varphi(y) \in Y$ for all $y \in Y$ and all $\varphi \in \mathcal{E}$. Equivalently, Y contains with every point y the *orbit $\mathcal{E}(y)$ of y* defined as

$$\mathcal{E}(y) := \{\varphi(y) \mid \varphi \in \mathcal{E}\}.$$

The closed semi-subgroup $\mathcal{E} \subseteq \text{End}(X)$ defines a linear subspace $\mathcal{D}_{\mathcal{E}} \subseteq \text{Vec}(X)$ as the image of the tangent space $T_{\text{id}} \mathcal{E}$ under Ξ :

$$\mathcal{D}_{\mathcal{E}} := \{\xi_A \mid A \in T_{\text{id}} \mathcal{E}\} \subseteq \text{Vec}(X).$$

The main point of this section is to relate the invariance under $\mathcal{D}_{\mathcal{E}}$ with the stability under the semigroup \mathcal{E} . A first and easy result is the following.

Proposition 3.2. *Let $\mathcal{E} \subseteq \text{End}(X)$ be a closed semi-subgroup. If $Y \subseteq X$ is a closed \mathcal{E} -stable subvariety, then Y is $\mathcal{D}_{\mathcal{E}}$ -invariant.*

Proof. Since Y is \mathcal{E} -invariant we have a morphism $\Phi: \mathcal{E} \times Y \rightarrow Y$ whose differential

$$d\Phi_{(\text{id}, y)}: T_{\text{id}} \mathcal{E} \oplus T_y Y \rightarrow T_y Y$$

sends $(A, 0)$ to $\xi_A(y)$, by formula $(*)$ above. Thus $\xi(y) \in T_y Y$ for all $\xi \in \mathcal{D}_{\mathcal{E}}$ which means that Y is $\mathcal{D}_{\mathcal{E}}$ -invariant. \square

We will see below that under stronger assumptions on \mathcal{E} the reverse implication also holds, i.e., a closed subset $Y \subseteq X$ is \mathcal{E} -stable if and only if it is $\mathcal{D}_{\mathcal{E}}$ -invariant.

Remark 3.3. We do not know what the structure of the subsets $\mathcal{E}(x) \subseteq X$ is. If \mathcal{E} is curve-connected (i.e. any two points of \mathcal{E} can be connected by an irreducible curve, see Definition 7.16(5)), then one can show that $\mathcal{E}(x)$ contains a set U which is open and dense in $\overline{\mathcal{E}(x)}$. But it is not clear whether $\mathcal{E}(x)$ is constructible.

3.2. The case of a vector space. In case of a vector space $X = V$ the situation is much simpler, because we can identify every tangent space $T_v V$ with V . In particular, vector fields $\xi \in \text{Vec}(V)$ correspond to morphisms $\xi: V \rightarrow V$. Choosing a basis of V we have

$$\xi = \sum_{i=1}^n p_i \frac{\partial}{\partial x_i} \text{ where } p_i := \xi x_i.$$

In this situation, the semigroup $\text{End}(V) = \mathcal{O}(V) \otimes V$ is a vector space, hence $T_{\text{id}} \text{End}(V) = \text{End}(V)$ in a canonical way, and

$$\Xi: \text{End}(V) = T_{\text{id}} \text{End}(V) \xrightarrow{\sim} \text{Vec}(V)$$

is the obvious isomorphism given as follows. In terms of coordinates an endomorphism φ has the form $\varphi = (p_1, \dots, p_n): \mathbb{k}^n \rightarrow \mathbb{k}^n$ where $p_i = \varphi^*(x_i)$, and the corresponding vector field $\xi := \Xi(\varphi)$ is given by $\xi = \sum_{i=1}^n p_i \frac{\partial}{\partial x_i}$.

The same formula holds for a semigroup $\mathcal{E} \subseteq \text{End}(V)$ which is a *linear* subspace. However, for a general closed semigroup $\mathcal{E} \subseteq \text{End}(V)$, we cannot identify \mathcal{E} with $T_{\text{id}}\mathcal{E}$, and so the formula above does not make sense. For example, if $\varphi \in \text{End}(V)$ is any endomorphism, then the semigroup $\mathcal{E} := \{\text{id}, \varphi, \varphi^2, \dots\}$ is discrete, hence $T_{\text{id}}\mathcal{E}$ is trivial, and so $\mathcal{D}_{\mathcal{E}}$ is also trivial.

The following result is crucial. We will identify $T_v V$ with V and thus consider the subspace $\mathcal{D}(v) \in T_v V$ as a subspace of V .

Lemma 3.4. *Let $\mathcal{E} \subseteq \text{End}(V)$ be a linear subspace which is a semigroup. Then*

$$\mathcal{E}(v) = M(v) = \mathcal{D}_{\mathcal{E}}(v) \text{ for all } v \in V.$$

In particular, a subset $Y \subseteq V$ is $\mathcal{D}_{\mathcal{E}}$ -invariant if and only if it is \mathcal{E} -stable. Moreover, we have $\mathcal{E}(w) = \mathcal{E}(v)$ for all w in an open neighborhood of v in $\mathcal{E}(v)$.

Proof. (a) We have seen in Proposition 3.2 that $\mathcal{E}(v)$ is $\mathcal{D}_{\mathcal{E}}$ -invariant, because it is stable under \mathcal{E} . Hence, $\mathcal{D}_{\mathcal{E}}(w) \subseteq T_w \mathcal{E}(v)$ for all $w \in \mathcal{E}(v)$.

(b) The evaluation map $\mu_v: \mathcal{E} \rightarrow V$ is linear with image $\mathcal{E}(v)$, hence $\mathcal{E}(v) \subseteq V$ is a linear subspace and $\mathcal{D}_{\mathcal{E}}(v) = T_v \mathcal{E}(v) = \mathcal{E}(v)$.

(c) By Lemma 2.7 there is an open neighborhood U_v of v in $\mathcal{E}(v)$ such that $\dim \mathcal{D}_{\mathcal{E}}(w) \geq \dim \mathcal{D}_{\mathcal{E}}(v)$ for all $w \in U_v$. Hence, $\mathcal{E}(v) = \mathcal{D}_{\mathcal{E}}(v) = \mathcal{D}_{\mathcal{E}}(w) = \mathcal{E}(w)$ for $w \in U_v$, by (a).

(d) It remains to prove the minimality, i.e. that $\mathcal{E}(v) = M(v)$. Let $Y \subseteq \mathcal{E}(v)$ be closed and $\mathcal{D}_{\mathcal{E}}$ -invariant with $v \in Y$. Then, for every $w \in U_v \cap Y$, we have $\mathcal{E}(v) = \mathcal{D}_{\mathcal{E}}(w) \subseteq T_w Y \subseteq \mathcal{E}(v)$. Hence, $\dim Y \geq \dim \mathcal{E}(v)$, and so $Y = \mathcal{E}(v)$. \square

Remark 3.5. If $\mathcal{E} \subseteq \text{End}(V)$ is as in the lemma above, then it contains the scalar multiplication $\mathbb{k} \cdot \text{id}$, and so $\mathcal{E}(v) \supset \mathbb{k}v$ for all $v \in V$. Therefore, every $\mathcal{D}_{\mathcal{E}}$ -invariant closed subvariety X is a closed cone, i.e., contains with every point $x \neq 0$ the line $\mathbb{k} \cdot x$, and every $\mathcal{D}_{\mathcal{E}}$ -invariant ideal is homogeneous.

3.3. Linear semigroups. One would like to extend the lemma above to a statement of the form that a subvariety $Y \subseteq X$ is stable under a closed semigroup $\mathcal{E} \subseteq \text{End}(X)$ if and only if it is $\mathcal{D}_{\mathcal{E}}$ -invariant where $\mathcal{D}_{\mathcal{E}}$ is the image of $T_{\text{id}}\mathcal{E}$ in $\text{Vec}(X)$. We do not know if such a result holds in general, but we can prove it for so-called *linear semigroups* $\mathcal{E} \subseteq \text{End}(X)$ which is sufficient for the applications we have in mind.

If $X \subseteq V$ is a closed subvariety, then $\text{End}(X) \subseteq \text{Mor}(X, V)$. Thus we can form linear combinations of endomorphisms of X , but in general the resulting morphism does not have its image in X .

Definition 3.6. A semi-subgroup $\mathcal{E} \subseteq \text{End}(X)$ is called *linear* if there is a closed embedding $X \hookrightarrow V$ into a vector space V such that the image of \mathcal{E} in $\text{Mor}(X, V)$ is a linear subspace.

Theorem 3.7. *Let X be an affine variety, and let $\mathcal{E} \subseteq \text{End}(X)$ be a linear semi-group.*

- (1) *For any $x \in X$ we have $\mathcal{E}(x) = M(x)$.*

- (2) The subsets $\mathcal{E}(x) \subseteq X$ are closed and isomorphic to vector spaces.
- (3) $T_x M(x) = T_x \mathcal{E}(x) = \mathcal{D}_{\mathcal{E}}(x)$ for all $x \in X$.

In particular, a closed subvariety $Y \subseteq X$ is $\mathcal{D}_{\mathcal{E}}$ -invariant if and only if it is \mathcal{E} -stable, i.e. it is a union of \mathcal{E} -orbits.

Proof. Choose a closed embedding $X \subseteq V$ such that $\mathcal{E} \subseteq \text{Mor}(X, V)$ is a linear subspace. Since the map $\text{End}(V) \rightarrow \text{Mor}(X, V)$ is linear and surjective there is a linear subspace $\tilde{\mathcal{E}} \subseteq \text{End}(V)$ whose image in $\text{Mor}(X, V)$ is \mathcal{E} . In particular, X is stable under $\tilde{\mathcal{E}}$ and so $\mathcal{D}_{\tilde{\mathcal{E}}} \subseteq \text{Vec}_X(V)$. The linearity of $\text{End}(V) \rightarrow \text{Mor}(X, V)$ implies that the image of $\mathcal{D}_{\tilde{\mathcal{E}}}$ under $\text{Vec}_X(V) \rightarrow \text{Vec}(X)$ is $\mathcal{D}_{\mathcal{E}}$, i.e. $\mathcal{D}_{\tilde{\mathcal{E}}}(x) = \mathcal{D}_{\mathcal{E}}(x)$ for all $x \in X$.

Now we apply Lemma 3.4 to $\tilde{\mathcal{E}}$ and find that $\mathcal{E}(x) = \tilde{\mathcal{E}}(x) = M(x)$, hence (1) and (2). Moreover, $T_x \mathcal{E}(x) = T_x \tilde{\mathcal{E}}(x) = \tilde{\mathcal{E}}(x) = \mathcal{D}_{\tilde{\mathcal{E}}}(x) = \mathcal{D}_{\mathcal{E}}(x)$, hence (3).

Finally, for a closed subvariety $Y \subseteq X \subseteq V$ the $\mathcal{D}_{\mathcal{E}}$ -invariance is the same as the $\mathcal{D}_{\tilde{\mathcal{E}}}$ -invariance, and Y is stable under $\tilde{\mathcal{E}}$ if and only if it is stable under \mathcal{E} . Hence the last claim follows also from the lemma. \square

If $\mathcal{E} \subseteq \text{End}(X)$ is a linear semigroup we define $d_{\mathcal{E}}(X) := \max_{x \in X} \dim \mathcal{E}(x)$. By our theorem above we have $d_{\mathcal{E}}(X) = d_{\mathcal{D}_{\mathcal{E}}}(X)$. We shall denote this common value simply by $d(X)$.

Corollary 3.8. *Let X be an irreducible variety and $\mathcal{E} \subseteq \text{End}(X)$ a linear semigroup. Then $X' := \{x \in X \mid \dim \mathcal{E}(x) = d(X)\}$ is open and dense in X , and the subsets $\mathcal{E}(x) \cap X'$ for $x \in X'$ form a partition of X' .*

Proof. The first part is Lemma 2.7. If $y \in \mathcal{E}(x)$, then $\mathcal{E}(y) \subseteq \mathcal{E}(x)$. Since, by the theorem above, the $\mathcal{E}(x)$ are vector spaces and $\dim \mathcal{E}(x) = \dim \mathcal{D}_{\mathcal{E}}(x)$, we have $\mathcal{E}(y) = \mathcal{E}(x)$ in case $y \in X'$. This proves the second claim. \square

Remark 3.9. If an algebraic group G acts on a variety X , then every element $A \in \text{Lie } G$ defines a vector field ξ_A . It is known that for a connected group G a closed subvariety $Y \subseteq X$ is G -stable if and only if Y is ξ_A -invariant for all $A \in \text{Lie } G$. A proof can be found in [Kra14, III.4.4, Corollary 4.4.7], and generalization to actions of connected ind-groups on affine varieties is given in [FK16]. Our main theorem above shows that a similar statement holds for linear semigroups.

4. G -SYMMETRY

4.1. G -equivariant endomorphisms. Now consider an action of an algebraic group G on the affine variety X . Then the induced actions of G on the coordinate ring $\mathcal{O}(X)$ and on the vector fields $\text{Vec}(X)$ are locally finite and rational, and the G -invariant vector fields $\text{Vec}_G(X)$ form an $\mathcal{O}(X)^G$ -module. Note that the (linear) action of G on $\text{Vec}(X)$ is given by $g\xi := dg \circ \xi \circ g^{-1}$ if we consider ξ as a section of the tangent bundle. If we regard ξ as a derivation δ of $\mathcal{O}(X)$, then $g\xi := (g^*)^{-1} \circ \delta \circ g^*$ where $g^*: \mathcal{O}(X) \rightarrow \mathcal{O}(X)$ is the comorphism of $g: X \rightarrow X$.

The action of G on $\text{End}(X)$ by conjugation induces a linear action on the tangent space $T_{\text{id}} \text{End}(X)$ which we denote by $g \mapsto \text{Ad } g$. It follows that the canonical map $\Xi: T_{\text{id}} \text{End}(X) \hookrightarrow \text{Vec}(X)$ is G -equivariant. In fact, one has the formula

$$\xi_{\text{Ad } g(A)}(gx) = dg \xi_A(x) \text{ for } A \in T_{\text{id}} \text{End}(X), g \in G \text{ and } x \in X.$$

This proves the first part of the following lemma.

Lemma 4.1. *We have $\xi(T_{\text{id}} \text{End}_G(X)) \subseteq \text{Vec}_G(X)$ with equality if X is a vector space V with a linear action of G .*

Proof. It remains to see that for a linear action of G on the vector space V we have $T_{\text{id}} \text{End}_G(V) = (T_{\text{id}} \text{End}(V))^G$. But this is clear, because $\text{End}(V)$ is a vector space, $\text{End}_G(V) = \text{End}(V)^G$ is a linear subspace, and $\Xi: T_{\text{id}} \text{End}(V) \xrightarrow{\sim} \text{Vec}(V)$ is a G -equivariant linear isomorphism. \square

4.2. G -symmetric subvarieties. We now come to the main notion of this paper, the G -symmetry of subvarieties. This was already discussed in the introduction.

Definition 4.2. Let X be an affine variety with an action of an algebraic group G . A closed subvariety $Y \subseteq X$ is called G -symmetric if Y is $\text{Vec}_G(X)$ -invariant, i.e., Y is parallel to all G -invariant vector fields ξ .

If V is a vector space with a linear action of the algebraic group G , then $\text{End}_G(V) \subseteq \text{End}(V)$ is a linear subspace and, by Lemma 4.1 above, the image of $T_{\text{id}} \text{End}_G(V)$ in $\text{Vec}(V)$ is the subspace $\text{Vec}_G(V)$ of G -invariant vector fields. Hence Theorem 3.7 implies the following result.

Theorem 4.3. *Let V be a vector space with a linear action of an algebraic group G . Then a closed subvariety $X \subseteq V$ is G -symmetric if and only if it is stable under $\text{End}_G(V)$.*

Example 4.4. Let V be a G -module, and assume that $V^G = \{0\}$. Define the *null cone*

$$\mathcal{N}_0 := \{v \in V \mid f(v) = 0 \text{ for all } f \in \mathcal{O}(V)^G \text{ such that } f(0) = 0\}.$$

Then $\mathcal{N}_0 \subseteq V$ is a closed G -symmetric subvariety.

Proof. We have $\mathcal{O}(V) = \mathbb{k} \oplus \mathfrak{m}_0$ where \mathfrak{m}_0 is the maximal ideal of $0 \in V$, and \mathcal{N}_0 is the zero set of \mathfrak{m}_0^G . Since V^G is fixed under every G -equivariant endomorphism φ of V , we get $\varphi^*(\mathfrak{m}_0^G) \subseteq \mathfrak{m}_0^G$, and so \mathcal{N}_0 is stable under $\text{End}_G(V)$. Now the claim follows from the theorem above. \square

4.3. Stabilizers. The next result deals with the relation between G -symmetric subvarieties and the G -action on X . We denote by $G_x \subseteq G$ the stabilizer of $x \in X$, and by $M(x)$ the minimal $\text{Vec}_G(X)$ -symmetric subvariety containing x (Lemma 2.5(4)).

Lemma 4.5. *Let X be an affine G -variety.*

- (1) *If $Y \subseteq X$ is a G -symmetric closed subvariety, then $gY \subseteq X$ is G -symmetric for all $g \in G$.*
- (2) *For $x \in X$ we have $\text{Vec}_G(X)(x) \subseteq (T_x X)^{G_x}$.*
- (3) *For $x \in X$ and $g \in G$ we have $gM(x) = M(gx)$, and so $gM(x) = M(x)$ for $g \in G_x$.*

Proof. (1) If ξ is a G -invariant vector field, then $d_g \xi(x) = \xi(gx)$ for $x \in X$, $g \in G$. This shows that $\xi(y) \in T_y Y$ if and only if $\xi(gy) \in T_{gy} gY$, and the claim follows.

(2) The formula in (1) shows that for a G -invariant vector field ξ we get $d_g \xi(x) = \xi(x)$ for $g \in G_x$. Hence $\xi(x) \in (T_x X)^{G_x}$.

(3) This follows from the minimality of $M(x)$. \square

In case of a linear action of G on a vector space V we get the following result.

Proposition 4.6. *Let V be a G -module.*

- (1) *For every closed subgroup $H \subseteq G$ the fixed point set V^H is G -symmetric.*
- (2) *For all $v \in V$ we have $M(v) = \text{End}_G(V)(v) \subseteq V^{G_v}$.*

Proof. (1) It is clear that V^H is stable under all G -equivariant endomorphisms, and so the claim follows from Theorem 4.3.

(2) By (1), V^{G_v} is G -symmetric and contains v , hence $M(v) \subseteq V^{G_v}$ by the minimality of $M(v)$. \square

Example 4.7. Let $G \rightarrow \text{GL}(V)$ be a diagonalizable representation of an algebraic group G . Then, for a generic $v \in V$, we have $\text{End}_G(V)(v) = V$. In particular, $d_{\text{End}_G(V)}(V) = \dim V$.

In fact, let $V = \bigoplus_{\chi \in \Omega} V_\chi$ be the decomposition into weight spaces where $\Omega \subseteq X(G)$ are those characters χ of G such that $V_\chi := \{v \in V \mid gv = \chi(g) \cdot v\}$ is nontrivial. Then $\text{End}_G(V)$ contains $\mathcal{L} := \bigoplus_{\chi \in \Omega} \mathcal{L}(V_\chi)$ where $\mathcal{L}(W)$ denotes the linear endomorphisms of the vector space W . It follows that for any $v = (v_\chi)_{\chi \in \Omega}$ such that $v_\chi \neq 0$ for all $\chi \in \Omega$ we have $\mathcal{L}(V) = V$, thus the claim.

4.4. Reductive groups. If X is an affine G -variety and $Y \subseteq X$ a closed and G -stable subvariety, then $\text{Vec}_Y(X) \subseteq \text{Vec}(X)$ is a G -submodule and the linear map $\rho: \text{Vec}_Y(X) \rightarrow \text{Vec}(Y)$ is G -equivariant and surjective. If Y is also G -symmetric, then $\rho(\text{Vec}_G(X)) \subseteq \text{Vec}_G(Y)$. But this might be a strict inclusion, i.e., not every G -invariant vector field on Y is obtained by restricting a G -invariant vector field from X (see Example 6.13 in section 6). However, if G is reductive, then we get $\rho(\text{Vec}_G(X)) = \text{Vec}_G(Y)$, because ρ is surjective and thus maps G -invariants onto G -invariants. This gives the following result.

Lemma 4.8. *Let V be a G -module and $X \subseteq V$ a closed G -stable and G -symmetric subvariety. If a closed subvariety $Y \subseteq X$ is G -symmetric with respect to the action of G on X , then it is also G -symmetric with respect to the action on V . If G is reductive, then the converse also holds.*

Example 4.9. Consider the adjoint representation of $\text{GL}_n = \text{GL}_n(\mathbb{k})$ on the matrices $M_n = M_n(\mathbb{k})$. It follows from classical invariant theory that $\text{End}_{\text{GL}_n}(M_n)$ is a free module over the invariants $\mathcal{O}(M_n)^{\text{GL}_n}$, with basis $(p_i: A \mapsto A^i \mid i = 0, \dots, n-1)$. Note that p_0 is the constant map $A \mapsto E$. It follows that the minimal symmetric subspaces $M(A)$ are given by

$$M(A) = \sum_{i=0}^{n-1} \mathbb{k} A^i.$$

In particular, a closed subset $Y \subseteq V$ is GL_n -symmetric if and only if, for any $A \in Y$, the vector space spanned by all powers $A^0 = E, A, A^2, \dots$ is contained in Y . Note that the minimal subsets $M(A) \subseteq M_n$ are exactly the *commutative unitary subalgebras* of $M_n(\mathbb{k})$ generated by one element.

Recall that a matrix A is *regular* if its centralizer $(\text{GL}_n)_A$ has dimension n which is the minimal dimension of a centralizer. Equivalently, the minimal polynomial of A coincides with the characteristic polynomial of A . The following is known.

- (1) A is regular if and only if $\dim M(A) = n$.

(2) For a regular matrix A one has $M(A) = (M_n)^{(\mathrm{GL}_n)_A}$.

An example of a closed G -symmetric subvariety is the nilpotent cone $\mathcal{N} \subseteq M_n$ consisting of all nilpotent matrices. It is also known that for a nilpotent matrix N all powers N^k are contained in the closure of the conjugacy class $C(N)$ of N , as well as their linear combinations. (In fact, $N' := \sum_{k \geq 0} a_k N^k$ is conjugate to N if $a_0 \neq 0$, because $\ker N'^j = \ker N^j$ for all j .) Hence these closures $\overline{C(N)}$ are G -symmetric as well.

In the example above we have $M(A) = (M_n)^{(\mathrm{GL}_n)_A}$ for a regular matrix A . This is an instance of the following general result which is due to PANYUSHEV [Pan02, Theorem 1]. For the convenience of the reader we give a short proof.

Proposition 4.10. *Let V be a G -module where G is reductive. If the closure \overline{Gv} of the orbit of v is normal and if $\mathrm{codim}_{\overline{Gv}}(\overline{Gv} \setminus Gv) \geq 2$, then $M(v) = V^{G_v}$.*

Proof. The assumptions on the orbit closure imply that $\mathcal{O}(\overline{Gv}) = \mathcal{O}(Gv)$. Let $w \in V^{G_v}$. We will show that there is a G -equivariant morphism $\varphi: V \rightarrow V$ such that $\varphi(v) = w$. Since $G_w \supseteq G_v$ there is a G -equivariant morphism $\mu: Gv \rightarrow V$ such that $\mu(v) = w$. The comorphism has the form $\mu^*: \mathcal{O}(V) \rightarrow \mathcal{O}(Gv) = \mathcal{O}(\overline{Gv})$, hence μ extends to a morphism $\tilde{\mu}: \overline{Gv} \rightarrow V$ which is again G -equivariant. Since G is reductive and $\overline{Gv} \subseteq V$ closed and G -stable, the morphism $\tilde{\mu}$ extends to a G -equivariant morphism $\varphi: V \rightarrow V$ with $\varphi(v) = w$. \square

4.5. Dense orbits. Let X be an irreducible affine variety, and let $\mathcal{E} \subseteq \mathrm{End}(X)$ be a semi-subgroup. An interesting question is whether \mathcal{E} has a dense orbit, i.e. whether there exists an $x \in X$ such that $\overline{\mathcal{E}(x)} = X$.

Lemma 4.11. *Let $\mathcal{E} \subseteq \mathrm{End}(X)$ be a linear semigroup. Then the following are equivalent.*

- (i) \mathcal{E} has a dense orbit in X .
- (ii) $d(X) = \dim X$.
- (iii) There exists an $x \in X$ such that $\mathcal{E}(x) = X$.
- (iv) One has $\mathcal{E}(x) = X$ for all x in an open dense subset of X .

If this holds, then X is a vector space.

Proof. If \mathcal{E} is a linear semigroup, then $\mathcal{E}(v) \subseteq V$ is a linear subspace and therefore closed in X . It is now clear that the first three statements are equivalent, and (iv) follows from (iii) and the last statement of Lemma 3.4. \square

Proposition 4.12. *Let G be a reductive group, and let V be a faithful G -module.*

- (1) *If the generic G -orbits in V are closed with trivial stabilizer, then $\mathrm{End}_G(V)$ has a dense orbit in V , i.e. $d(V) = \dim V$.*
- (2) *If G is semisimple and $d(V) = \dim V$, then the generic G -orbits in V are closed with trivial stabilizer.*

Proof. Set $\mathcal{E} := \mathrm{End}_G(V)$.

(1) If the orbit Gv is closed and G_v trivial, then $\mathcal{E}(v) = V$ by Proposition 4.10.

(2) If G is semisimple, then the generic G -orbits in V are closed. If $d(V) = \dim V$, then, by the lemma above, we have $\mathcal{E}(v) = V$ for all v from a dense open subset $U \subseteq V$. Since $\mathcal{E}(v) \subseteq V^{G_v}$ and since the action is faithful, we see that G_v is trivial for all $v \in U$, i.e. the generic stabilizer is trivial. \square

Remark 4.13. Example 4.7 shows that the assumption in (2) that G is semisimple is necessary.

5. FIRST INTEGRALS

5.1. The field of first integrals. Let X be an irreducible affine variety, and let $\mathcal{D} \subseteq \text{Vec}(X)$ be a linear subspace.

Definition 5.1. A *first integral* of \mathcal{D} is a rational function $f \in \mathbb{k}(X)$ with the property that $\xi f = 0$ for all $\xi \in \mathcal{D}$. If X is a G -variety and $\mathcal{D} := \text{Vec}_G(X)$, then a first integral of \mathcal{D} will be called a *first integral for the G -action on X* .

It is easy to see that the first integrals of \mathcal{D} form a subfield of $\mathbb{k}(X)$ which we denote by $\mathcal{F}_{\mathcal{D}}(X)$. If $\mathcal{D} = \text{Vec}_G(X)$, then we write $\mathcal{F}_G(X)$ instead of $\mathcal{F}_{\text{Vec}_G(X)}(X)$.

From now on assume that X is an irreducible affine variety, and that $\mathcal{D} \subseteq \text{Vec}(X)$ is a linear subspace. We want to show that the first integrals are the rational functions on a certain “quotient” of the variety X .

Lemma 5.2. *Let $f \in \mathbb{k}(X)$ be a rational function.*

- (1) *Assume that there is an open dense $U \subseteq X$ where f is defined and has the property that f is constant on $M(x) \cap U$ for all $x \in U$. Then f is a first integral of \mathcal{D} .*
- (2) *Assume that f is a first integral of \mathcal{D} . If f is defined in $x \in X$ and if $T_x M(x) = \mathcal{D}(x)$, then f is constant on $M(x)$.*

Proof. (1) Since $M(x)$ is \mathcal{D} -invariant we have $\xi(x) \in T_x M(x)$ for all $x \in U$ and all $\xi \in \mathcal{D}$. Hence $(\xi f)(x) = \xi(x)f = \xi(x)f|_{M(x) \cap U} = 0$, because $f|_{M(x) \cap U}$ is constant, and so $\xi f = 0$ for all $\xi \in \mathcal{D}$.

(2) There is $d \geq 0$ such that $\dim \mathcal{D}(y) \leq d$ for all $y \in M(x)$, with equality on a dense open set $M' \subseteq M(x)$ (Lemma 2.7). In particular, $\dim M(x) \leq \dim T_x M(x) = \dim \mathcal{D}_x \leq d$. On the other hand, $\mathcal{D}_y \subseteq T_y M(x)$ for all $y \in M(x)$. We can assume that M' consists of smooth point of $M(x)$. Then, for every $y \in M'$, we get $d = \dim \mathcal{D}_y \leq \dim T_y M(x) = \dim M(x)$. Hence $d = \dim M(x)$, and so $T_y M(x) = \mathcal{D}(y)$ for all $y \in M'$. Since f is defined in x , it is defined in a dense open set $M'' \subseteq M'$. But then $f|_{M''}$ is constant, because $\delta f = 0$ for all $u \in M''$ and all $\delta \in T_u M(x)$. \square

Remark 5.3. If $\mathcal{E} \subseteq \text{End}(X)$ is a linear semigroup and $\mathcal{D} := \mathcal{D}_{\mathcal{E}}$, then a rational function $f \in \mathbb{k}(X)$ defined on an open set $U \subseteq X$ is a first integral for \mathcal{D} if and only if f is constant on $\mathcal{E}(x) \cap U$ for all $x \in U$. This follows from the lemma above, because in this case we have $\mathcal{E}(x) = M(x)$ and $T_x M(x) = \mathcal{D}(x)$ for all $x \in X$, by Theorem 3.7.

Now choose a closed embedding $X \subseteq V$ into a vector space V . We know from Lemma 2.7 that $X' := \{x \in X \mid \dim \mathcal{D}(x) = d(X)\}$ is open and dense in X . Consider the map

$$\pi: X' \rightarrow \text{Gr}_{d(X)}(V) \text{ given by } \pi(x) := \mathcal{D}(x) \subseteq T_x X \subseteq V.$$

Lemma 5.4. *The map $\pi: X' \rightarrow \text{Gr}_{d(X)}(V)$ is a morphism of varieties.*

Proof. We will use the Plücker-embedding $\text{Gr}_d(V) \hookrightarrow \mathbb{P}(\bigwedge^d V)$, $d := d(X)$. For $x \in X'$ choose $\xi_1, \dots, \xi_d \in \mathcal{D}$ such that $\xi_1(x), \dots, \xi_d(x)$ is a basis of $\mathcal{D}(x)$. Then $\mathcal{D}(x) =$

$\xi_1(x) \wedge \xi_2(x) \wedge \cdots \wedge \xi_d(x) \in \bigwedge^d V$. It follows that there is an open neighborhood $U_x \subseteq X'$ of x such that $\xi_1(u), \dots, \xi_d(u)$ is a basis of $\mathcal{D}(u)$ for all $u \in U_x$. Since $\pi(u) = [\xi_1(u) \wedge \cdots \wedge \xi_d(u)] \in \mathbb{P}(\bigwedge^d V)$ we see that $\pi|_{U_x}$ is a morphism, and the claim follows. \square

5.2. The quotient mod \mathcal{E} . Let $\mathcal{E} \subseteq \text{End}(V)$ be a linear semigroup, and let $\mathcal{D}_{\mathcal{E}} \subseteq \text{Vec}(V)$ denote the image of $T_{\text{id}}\mathcal{E} = \mathcal{E}$. Let $X \subseteq V$ be a closed irreducible \mathcal{E} -stable subvariety. Under these assumptions we have $\mathcal{E}(x) = M(x) = \mathcal{D}_{\mathcal{E}}(x) \subseteq V$ for all $x \in X$ (Lemma 3.4). As above, define

$$X' := \{x \in X \mid \dim \mathcal{E}(x) = d(X)\},$$

and consider the morphism $\pi: X' \rightarrow \text{Gr}_{d(X)}(V)$, $x \mapsto \mathcal{E}(x) \subseteq T_x X \subseteq V$.

Proposition 5.5. (1) For all $x \in X'$ we have $\pi^{-1}(\pi(x)) = \mathcal{E}(x) \cap X'$.

(2) π induces an isomorphism $\pi^*: \mathbb{k}(\overline{\pi(X')}) \xrightarrow{\sim} \mathcal{F}_{\mathcal{D}_{\mathcal{E}}}(X)$.

(3) We have $\text{tdeg}_{\mathbb{k}} \mathcal{F}_{\mathcal{D}_{\mathcal{E}}}(X) = \dim X - d(X) = \dim \pi(X')$.

(4) $\mathcal{F}_{\mathcal{D}_{\mathcal{E}}}(X) = \mathbb{k}$ if and only if $d(X) = \dim X$, and then $X \subseteq V$ is a linear subspace.

We will use the notion $X//\mathcal{E}$ for the closure $\overline{\pi(X')}$ reflecting the fact that the map π can be regarded as the “quotient” under the action of the semigroup \mathcal{E} of endomorphisms.

Proof. (1) For $y \in \mathcal{E}(x) \cap X'$ we have $\mathcal{E}(y) = \mathcal{E}(x)$, hence $\pi(y) = \pi(x)$. If $y \in X' \setminus \mathcal{E}(x)$, then $\mathcal{E}(y) \neq \mathcal{E}(x)$ and so $\pi(y) \neq \pi(x)$.

(2) By Remark 5.3 a rational function $f \in \mathbb{k}(X)$ defined on an open set $U \subseteq X'$ is a first integral if and only if it is constant on the subsets $\mathcal{E}(x) \cap U$ for all $x \in U$. We can assume that $\pi(U) \subseteq \text{Gr}_{d(X)}(V)$ is locally closed and smooth and that $\pi: U \rightarrow \pi(U)$ is smooth. Then it is a well-known fact that $\pi^*(\mathcal{O}(\pi(U))) \subseteq \mathcal{O}(U)$ are the regular functions on U which are constant on the fibers.

(3) This is clear.

(4) If $d(X) = \dim X$, then $X = \mathcal{E}(x)$ for a generic $x \in X$ (Lemma 4.11), and so X is a linear subspace of V . \square

Corollary 5.6. If $X \subseteq V$ is not a linear subspace, then there exist non-constant first integrals.

Note that if X is smooth, then it is a linear subspace, because X is a closed cone, see Remark 3.5.

Example 5.7. Let $X \subseteq V$ be a closed cone, and let $\mathcal{E} := \mathbb{k} \cdot \text{id}_V \subseteq \text{End}(V)$. Then $\mathcal{E}(x) = \mathbb{k}x$ for all $x \in X$, hence $X//\mathcal{E} = \mathbb{P}(X)$ and $\mathcal{F}_{\mathcal{D}_{\mathcal{E}}}(X) = \mathbb{k}(\mathbb{P}(X))$.

5.3. The symmetric case. Assume that V is a representation of an algebraic group and that $\mathcal{E} := \text{End}_G(V)$, hence $\mathcal{D}_{\mathcal{E}} = \text{Vec}_G(V)$. Then, for every G -stable and G -symmetric closed irreducible subvariety $X \subseteq V$, the open subset $X' \subseteq X$ is G -stable and the morphism $\pi: X' \rightarrow \text{Gr}_{d(X)}(V)$ is G -equivariant. In particular, $\pi^*: \mathbb{k}(\overline{\pi(X')}) \xrightarrow{\sim} \mathcal{F}_G(X)$ is a G -equivariant isomorphism. It follows that for any $x \in X'$ we have

$$G_{\pi(x)} = \text{Norm}_G(\mathcal{E}(x))$$

where $\text{Norm}_G(W)$ denotes the normalizer in G of the subspace $W \subseteq V$.

Lemma 5.8. (1) For $x \in X'$ we have

$$\mathrm{tdeg} \mathcal{F}_G(X) \geq \mathrm{tdeg} \mathcal{F}_G(X)^G + \dim G - \dim \mathrm{Norm}_G(\mathcal{E}(x))$$

with equality on a dense open set $U \subseteq X'$.

(2) If $\mathcal{F}_G(X)^G = \mathbb{k}$, then $\mathcal{F}_G(X)$ is G -isomorphic to $\mathbb{k}(G/\mathrm{Norm}_G(\mathcal{E}(x)))$ for any x in a dense open set of X' .

Proof. (1) By ROSENBLICHT's theorem (see [Spr89, Satz 2.2]) there is an open dense G -stable subset $O \subseteq \pi(X')$ which admits a geometric quotient $q: O \rightarrow O/G$. In particular, the fibers of q are G -orbits and have all the same dimension. Hence $\mathrm{tdeg} \mathcal{F}_G(X) = \dim O = \dim O/G + \dim Gu$ for $u \in O$, and $\mathbb{k}(O/G) = \mathbb{k}(O)^G = \mathcal{F}_G(X)^G$. If $u = \pi(x)$, then $Gu = \mathrm{Norm}_G(\mathcal{E}(x))$ and so

$$\mathrm{tdeg} \mathcal{F}_G(X) = \mathrm{tdeg} \mathcal{F}_G(X)^G + \dim G - \dim \mathrm{Norm}_G(\mathcal{E}(x))$$

for all $x \in U := \pi^{-1}(O)$. Since $\dim Gu$ is maximal for $u \in O$ the claim follows.

(2) If $\mathcal{F}_G(X)^G = \mathbb{k}$, then, as a consequence of ROSENBLICHT's theorem, G has a dense orbit Gu in $\pi(X')$ and so $\mathcal{F}_G(X) = \mathbb{k}(Gu)$. If $u = \pi(x)$, then $Gu \simeq G/\mathrm{Norm}_G(\mathcal{E}(x))$, and the claim follows. \square

Remark 5.9. Note that $\mathcal{F}_G(X)^G = \mathbb{k}$ if and only if $G\mathcal{E}(x)$ is dense in X for a generic $x \in X$, or, equivalently, $\dim X = d(X) + \dim G - \dim \mathrm{Norm}_G(\mathcal{E}(x))$ for a generic $x \in X$.

Example 5.10. Consider the adjoint representation of GL_2 on M_2 . Then $M'_2 = M_2 \setminus \mathbb{k}I_2$ where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and the morphism π is equal to the composition

$$\pi: M'_2 \twoheadrightarrow (M_2/\mathbb{k}I_2) \setminus \{0\} \twoheadrightarrow \mathbb{P}(M_2/\mathbb{k}I_2).$$

Choosing the basis $\overline{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \overline{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \overline{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$ of $M_2/\mathbb{k}I_2$, the pullbacks of the coordinate functions are $b, c, \frac{a-d}{2}$, and so $\mathcal{F}_{\mathrm{GL}_2}(M_2) = \mathbb{k}(\frac{a-d}{b}, \frac{a-d}{c})$.

Example 5.11. For the adjoint representation of GL_n on M_n we claim that GL_n has a dense orbit in $\pi(M'_n)$. In fact, let $S \in M_n$ be a generic diagonal matrix. Then the span $\mathcal{E}(S) = \sum_{i=0}^{n-1} \mathbb{k}S^i$ has dimension n , hence it is the subspace of diagonal matrices, and so $\mathrm{GL}_n \mathcal{E}(S) \subseteq M_n$ is the dense subset of all diagonalizable matrices. Moreover, the normalizer of $\mathcal{E}(S)$ is equal to N , the normalizer of the diagonal torus $T \subseteq \mathrm{GL}_n$, and so $\mathcal{F}_{\mathrm{GL}_n}(M_n) \simeq \mathbb{k}(\mathrm{GL}_n/N)$.

Example 5.12. The previous example carries over to the adjoint representation of an arbitrary semisimple group G on its Lie algebra $\mathfrak{g} := \mathrm{Lie} G$. If $s \in \mathfrak{g}$ is a regular semisimple element, then the orbit Gs is closed and the stabilizer of s is a maximal torus T . This implies by Proposition 4.10 that $\mathcal{E}(s) = \mathfrak{g}^T = \mathrm{Lie} T$ which is a toral subalgebra of \mathfrak{g} . Again, $G\mathcal{E}(s) \subseteq \mathfrak{g}$ is the dense set of semisimple elements of \mathfrak{g} , and the normalizer of $\mathcal{E}(s)$ is equal to N , the normalizer of T in G . Hence $\mathcal{F}_G(\mathrm{Lie} G) \simeq \mathbb{k}(G/N)$.

In the examples above there are no G -invariant first integrals: $\mathcal{F}_G(X)^G = \mathbb{k}$. This is not always the case as the next two examples show. However, it holds for a representation of a reductive group G in case the generic fiber of the quotient map contains a dense orbit (Proposition 5.18).

Example 5.13. Let $U = \left\{ \begin{bmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{bmatrix} \mid a, b, c \in \mathbb{k} \right\} \subseteq \mathrm{GL}_3(\mathbb{k})$ be the unipotent group of upper triangular matrices, and consider the adjoint representation of U on its Lie

algebra $\mathfrak{u} := \text{Lie } U = \left\{ \begin{bmatrix} 0 & x & y \\ 0 & z & 0 \end{bmatrix} \mid x, y, z \in \mathbb{k} \right\}$. For $u = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \end{bmatrix} \in U$ and $v = \begin{bmatrix} 0 & x & y \\ 0 & z & 0 \end{bmatrix} \in \mathfrak{u}$ we find

$$(**) \quad \text{Ad}(u)v = uvu^{-1} = \begin{bmatrix} 0 & x & -cx + y + az \\ 0 & & z \\ & & 0 \end{bmatrix}$$

which shows that the fixed points are $\mathfrak{u}^U = \mathbb{k} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and the other orbits are the parallel lines $\text{Ad}(U) \begin{bmatrix} 0 & x & y \\ 0 & z & 0 \end{bmatrix} = \begin{bmatrix} 0 & x & 0 \\ 0 & z & 0 \end{bmatrix} + \mathfrak{u}^U$. It follows that the invariant ring is given by $\mathcal{O}(\mathfrak{u})^U = \mathbb{k}[x, z]$. We have an exact sequence of U -modules

$$0 \rightarrow \mathfrak{u}^U \hookrightarrow \mathfrak{u} \xrightarrow{p} \mathbb{k}^2 \rightarrow 0 \quad \text{where } p\left(\begin{bmatrix} 0 & x & y \\ 0 & z & 0 \end{bmatrix}\right) := (x, z).$$

We claim that the covariants $\mathcal{E} := \text{Cov}(\mathfrak{u}, \mathfrak{u})$ are generated as a $\mathcal{O}(\mathfrak{u})^U$ -module by $\text{id}_{\mathfrak{u}}$ and $\varphi_0: \begin{bmatrix} 0 & x & y \\ 0 & z & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. This implies that $\mathcal{E}(v) = \mathbb{k}v + \mathfrak{u}^U$ for $v \in \mathfrak{u} \setminus \mathfrak{u}^U$, hence $d(\mathfrak{u}) = 2$ and $\mathfrak{u}' = \mathfrak{u} \setminus \mathfrak{u}^U$. It follows that

$$\mathfrak{u} // \mathcal{E} = \mathbb{P}(\mathfrak{u}/\mathfrak{u}^U) \xrightarrow{\sim} \mathbb{P}^1.$$

In particular, the action of U on the quotient is trivial, and so

$$\mathcal{F}_U(\mathfrak{u}) = \mathcal{F}_U(\mathfrak{u})^U = \mathbb{k}\left(\frac{x}{z}\right).$$

In order to prove the claim, let $\varphi: \mathfrak{u} \rightarrow \mathfrak{u}$ be a covariant,

$$\varphi\left(\begin{bmatrix} 0 & x & y \\ 0 & z & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & p(x, y, z) & q(x, y, z) \\ 0 & & r(x, y, z) \end{bmatrix} \quad \text{where } p, q, r \in \mathcal{O}(\mathfrak{u}) = \mathbb{k}[x, y, z].$$

Then, by $(**)$, we get for $a, b, c \in \mathbb{k}$

$$(1) \quad q(x, -cx + y + az, z) = -c \cdot p(x, y, z) + q(x, y, z) + a \cdot r(x, y, z).$$

This shows that q is linear in y , i.e. $q(x, y, z) = q_0(x, z) + q_1(x, z)y$, and so

$$(2) \quad q(x, -cx + y + az, z) = q_0(x, y) + q_1(x, z)(-cx + y + az) = q_0 - c \cdot q_1 x + q_1 y + a \cdot q_1 z.$$

Comparing (2) with (1) we get

$$(3) \quad p = q_1 x, \quad q = q_0 + q_1 y, \quad r = q_1 z,$$

hence $\varphi = q_1 \text{id}_{\mathfrak{u}} + q_0 \varphi_0$, as claimed.

Example 5.14. Let G be a reductive group and V an irreducible G -module. If the connected component of the center $Z(G)^0$ acts nontrivially, then $\text{End}_G(V) = \mathbb{k} \text{id}_V$. Hence, by Example 5.7, we get $V // \text{End}_G(V) \simeq \mathbb{P}(V)$, $\mathcal{F}_G(V) = \mathbb{k}(\mathbb{P}(V))$, and $\mathcal{F}_G(V)^G = \mathbb{k}(\mathbb{P}(V // (G, G)))$.

(In order to see that $\text{End}_G(V) = \mathbb{k} \text{id}_V$ we just remark that the G -module V^* occurs only once in $\mathcal{O}(V)$, namely in degree 1. In fact, $Z(G)^0$ acts on V via a character χ , and thus via χ^{-d} on the homogeneous functions $\mathcal{O}(V)_d$ of degree d .)

This example generalizes to the situation where V is a reducible G -module such that the characters of $Z(G)^0$ on the irreducible components of V are linearly independent.

Example 5.15. Let V be an irreducible representation of a reductive group G . For the orbit $O_{\min} \subseteq V$ of highest weight vectors we have $\overline{O_{\min}} = O_{\min} \cup \{0\}$, and $\overline{O_{\min}}$ is normal with rational singularities (see [Hes79]). Clearly, $\overline{O_{\min}}$ is G -symmetric, i.e. stable under all G -equivariant endomorphisms of V . We claim that $\mathcal{E} := \text{End}_G(\overline{O_{\min}}) = \mathbb{k} \cdot \text{id}$. In fact, if $v \in V$ is a highest weight vector, then the G -orbit of $[v] \in \mathbb{P}(V)$ is closed, and thus the normalizer P of $[v]$ is a parabolic subgroup. Hence P is the normalizer of G_v in G , and so $P/G_v = \mathbb{k}^*$. Since, $\text{Aut}_G(\overline{O_{\min}}) = \text{Aut}_G(O_{\min}) \simeq P/G_v = \mathbb{k}^*$ the claim follows.

As a consequence we get $\overline{O_{\min}}' = O_{\min}$, $\overline{O_{\min}}/\mathcal{E} = O_{\min}/\mathbb{k}^* = \mathbb{P}(\overline{O_{\min}}) \subseteq \mathbb{P}(V)$, and so $\mathbb{P}(\overline{O_{\min}})$ is the closed orbit of highest weight vectors in $\mathbb{P}(V)$. In particular, $\mathcal{F}_G(\overline{O_{\min}}) = \mathbb{k}(\mathbb{P}(O_{\min}))$, and $\mathcal{F}_G(\overline{O_{\min}})^G = \mathbb{k}$.

5.4. First integrals for reductive groups. Let G be a reductive group, and let X be an irreducible G -variety. Denote by $q: X \rightarrow X//G$ the quotient. Then LUNA's slice theorem (see [Lun73, pp. 97–98]) implies the existence of a *principal isotropy group* $H \subseteq G$. This means the following:

- (1) If $Gx \subseteq X$ is a closed orbit, then G_x contains a conjugate of H .
- (2) The set $(X//G)_{\text{pr}}$ of points $\xi \in X//G$ such that the closed orbit in the fiber $q^{-1}(\xi)$ is G -isomorphic to G/H is open and dense in $X//G$.

It follows that every closed orbit contains a fixed point of H , hence $\pi(X^H) = X//G$.

The open dense subset $(X//G)_{\text{pr}}$ of $X//G$ is called the *principal stratum*, and the closed orbits over the principal stratum are the *principal orbits*. If the action on X is *stable*, i.e. if the generic orbits of X are closed, then the principal orbits are generic.

Theorem 5.16. *Let G be reductive, V a G -module, and let $X \subseteq V$ be a G -stable and G -symmetric irreducible closed subvariety. Assume that the generic orbit of X is closed, with principal isotropy group $H \subseteq G$. Then $\mathcal{F}_G(X) = \mathbb{k}(G/N)$ where $N := \text{Norm}_G(H)$. In particular, $\mathcal{F}_G(X)^G = \mathbb{k}$.*

Proof. By assumption, the orbit Gx is principal for a generic $x \in X^H$. The minimal invariant subset $M(x) \subseteq X$ is also minimal invariant as a subset of V (Lemma 4.8). Hence, $M(x) = V^H$ by Proposition 4.10. Since $M(x) \subseteq X^H \subseteq V^H$, we finally get $M(x) = X^H = V^H$. As we have seen above, GX^H contains all closed orbits, and in particular all $M(y)$ for y in the dense open set of principal orbits. This implies that G has a dense orbit in $X//\text{End}_G(X)$. Since the stabilizer of the image $\pi(V^H)$ is the normalizer $\text{Norm}_G(V^H)$, it remains to see $\text{Norm}_G(V^H) = \text{Norm}_G(H)$. Since $g(V^H) = V^{gHg^{-1}}$ we get $V^H = V^{H \cap gHg^{-1}}$ for any $g \in \text{Norm}(V^H)$, hence $H = gHg^{-1}$, because the stabilizer of a generic elements from V^H is H . \square

Since the generic orbits in a representation of a semisimple group are closed, we get the following consequence.

Corollary 5.17. *Let V be a representation of a semisimple group G , and let $H \subseteq G$ be the principal isotropy group. Then $\mathcal{F}_G(V) = \mathbb{k}(G/N)$ where $N := \text{Norm}_G(H)$, and $\mathcal{F}_G(V)^G = \mathbb{k}$. In particular, $\mathcal{F}_G(V) = \mathbb{k}$ if the principal isotropy group is trivial.*

Note that for a “generic” representation of a semisimple group G the principal isotropy group is trivial, hence there are no nonconstant first integrals. The irreducible representations of simple groups with a nontrivial principal isotropy group have been classified ([AVE67], cf. [VP94, §7]).

The fact that there are no G -invariant first integrals is a consequence of the following slightly more general result.

Proposition 5.18. *Let V be representation of a reductive group G . Assume that the generic fiber of the quotient map $q: V \rightarrow V//G$ contains a dense orbit $O \simeq G/K$, or, equivalently, $\mathbb{k}(V)^G$ is the field of fractions of $\mathcal{O}(V)^G$. Then $\mathcal{F}_G(V) \simeq \mathbb{k}(G/\text{Norm}_G(K))$ and $\mathcal{F}_G(V)^G = \mathbb{k}$.*

Proof. Let F be a fiber of the quotient map q over the principal stratum, and let $O \subseteq F$ be the dense orbit. Consider the morphism $\pi: V' \rightarrow V//\text{End}_G(V) \subseteq \text{Gr}_d(V)$, $d := d_{\text{End}_G(V)}(V)$. We claim that $O \subseteq V'$, that $\overline{\pi(O)} = \overline{\pi(V')} = V//\text{End}_G(V)$, and that the image of O under π is $G/\text{Norm}_G(K)$. This proves the proposition.

LUNA's slice theorem tells us that the fibers of the quotient map over the principal stratum are G -isomorphic. This implies that $\text{End}_G(V)$ acts transitively on the set of these fibers (see the argument in the proof of Proposition 4.10), hence $\overline{\pi(V')} = \overline{\pi(F \cap V')}$. Since $F \cap V'$ is open and G -stable, we have $O \subseteq V'$. If $\varphi \in \text{End}_G(V)$ and $\varphi(v) \in O$ for some $v \in O$, then $\varphi(O) = O$, and $\varphi|_O$ is a G -equivariant automorphism. Hence, $\pi(O) \simeq O/\text{Aut}_G(O) \simeq G/\text{Norm}_G(K)$, and the claims follow. \square

6. ACTIONS OF SL_2

6.1. Representations. The standard representation of SL_2 on $V := \mathbb{k}^2$ defines a linear action on the coordinate ring $\mathcal{O}(V) = \mathbb{k}[x, y]$ given by $gf(v) := f(g^{-1}v)$. It is well-known that the homogeneous components $V_d := \mathbb{k}[x, y]_d$, $d = 0, 1, 2, \dots$, represent all irreducible representations of SL_2 , i.e. all simple SL_2 -modules. For these representations we have the following result.

Proposition 6.1. *Define $\mathcal{E}_d := \text{End}_{\text{SL}_2}(V_d)$.*

- (1) $\mathcal{E}_1 = \mathbb{k}\text{id}_{V_1}$, hence $d(V_1) = 1$. Moreover, we have $V'_1 = V_1 \setminus \{0\}$, $V_1//\mathcal{E}_1 = \mathbb{P}(V_1)$ and $\mathcal{F}_{\text{SL}_2}(V_1) \simeq \mathbb{k}(\text{SL}_2/B)$.
- (2) $\mathcal{E}_2 = J\text{id}_{V_2}$ where $J := \mathcal{O}(V_2)^{\text{SL}_2} = \mathbb{k}[D]$ is the ring of invariants where D is the discriminant. Hence $d(V_2) = 1$, $V'_2 = V_2 \setminus \{0\}$, $V_2//\mathcal{E}_2 = \mathbb{P}(V_2)$ and $\mathcal{F}_{\text{SL}_2}(V_2) \xrightarrow{\sim} \mathbb{k}(\text{SL}_2/N)$.
- (3) For $d \geq 3$, we have $\mathcal{E}_d(f) = V_d$ for a generic $f \in V_d$, hence $d(V_d) = \dim V_d$ and $\mathcal{F}_{\text{SL}_2}(V_d) = \mathbb{k}$.

Proof. (1) The coordinate ring $\mathcal{O}(V_1)$ contains every irreducible representation exactly once. Hence every covariant of V_1 is a multiple of the identity, and so $\mathcal{E}(v) = \mathbb{k}v$ for all $v \in V_1$. Now the claims follow.

(2) This follows immediately from Example 5.10.

(3) For $d \geq 3$ the stabilizer of a generic element $f \in V_d$ is trivial for d odd and $\pm I_2$ for d even. Hence, $\mathcal{E}(f) = V_d$ by Proposition 4.10, and the claims follow. \square

Remark 6.2. An SL_2 -equivariant morphism $\varphi: V \rightarrow W$ between two SL_2 -modules is called a *covariant*. Every covariant is a sum of homogeneous covariants:

$$\text{Cov}(V, W) = \bigoplus_{j \in \mathbb{N}} \text{Cov}(V, W)_j.$$

Moreover, $\text{End}_{\text{SL}_2}(V) = \text{Cov}(V, V)$ is a module under $\mathcal{O}(V)^{\text{SL}_2}$ where the module structure is given by $f\varphi(v) := f(v) \cdot \varphi(v)$.

6.2. The nullcone $\mathcal{N}(V)$. A very interesting object in this setting is the *nullcone* $\mathcal{N}(V) \subseteq V$ of a representation V of SL_2 which is defined in the following way. Denote by $q: V \rightarrow V//\mathrm{SL}_2$ the quotient morphism, i.e., $V//\mathrm{SL}_2 = \mathrm{Spec} \mathcal{O}(V)^{\mathrm{SL}_2}$ and q is induced by the inclusion $\mathcal{O}(V)^{\mathrm{SL}_2} \subseteq \mathcal{O}(V)$. Then $\mathcal{N}(V) := q^{-1}(q(0))$, or equivalently, $\mathcal{N}(V)$ is the zero set of all homogeneous invariants of positive degree. In case $V = V_d$ the elements from $\mathcal{N}(V_d)$ are classically called *nullforms*. One has the following description. Denote by $T \subseteq \mathrm{SL}_2$ the diagonal torus, and define the *weight spaces*

$$V[i] := \{f \in V \mid \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} f = t^i f \text{ for all } t \in \mathbb{k}^*\} \text{ for } i \in \mathbb{N}.$$

Since the representation of T is completely reducible we have $V = \bigoplus_j V[j]$. For $V = V_d$ we get $V_d = \bigoplus_{i=0}^d V_d[d-2i]$, and the weight spaces are one-dimensional. Note that $\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} x = t^{-1}x$, and $\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} y = ty$, and so

$$V_d[d-2i] = \mathbb{k}x^i y^{d-i}.$$

Lemma 6.3. *The following statements for a form $f \in V_d$ are equivalent.*

- (i) f is a nullform, i.e. $f \in \mathcal{N}(V_d)$.
- (ii) There is a one-parameter subgroup $\lambda: \mathbb{k}^* \rightarrow \mathrm{SL}_2$ such that $\lim_{t \rightarrow 0} \lambda(t)f = 0$.
- (iii) f is in the SL_2 -orbit of an element from $V_d^+ := \bigoplus_{i>0} V_d[i] \subseteq V_d$.
- (iv) f contains a linear factor with multiplicity $> \frac{d}{2}$.

Proof. (a) The equivalence of (i) and (ii) is a consequence of the famous HILBERT-MUMFORD-Criterion and holds for any representation of a reductive group.

(b) (ii) and (iii) are equivalent, because every one-parameter subgroup of SL_2 is conjugate to a one-parameter subgroup of T . This holds for any representation of SL_2 .

(c) The equivalence of (iii) and (iv) is clear, because V_d^+ are the forms which contain y with multiplicity $> \frac{d}{2}$. \square

Let V be a representation of SL_2 . If $\varphi \in \mathrm{End}_{\mathrm{SL}_2}(V)$ is homogeneous of degree k , then $\varphi(V[j]) \subseteq V[k+j]$. It follows that $\varphi(\bigoplus_{j \geq j_0} V[j]) \subseteq \bigoplus_{j \geq k+j_0} V[j]$. In particular, the subspaces $\bigoplus_{j \geq j_0} V[j]$ are G -symmetric for any $j_0 \geq 0$, because any endomorphism is a sum of homogeneous endomorphisms (Remark 6.2). Since every element $f \in \mathcal{N}(V)$ is SL_2 -equivalent to an element from $V^+ := \bigoplus_{j>0} V[j]$ it suffices to study the SL_2 -symmetric subspaces of V^+ . Note that such a subspace is T -stable.

6.3. Covariants. We will now construct some special covariants for the binary forms V_d . Let $\varphi: V_d \rightarrow \mathrm{End}(V_d)$ and $\psi: V_d \rightarrow V_d$ be homogeneous covariants. Then we define covariants $\Phi_s = \Phi_s(\varphi, \psi) \in \mathrm{End}_{\mathrm{SL}_2}(V_d)$ by

$$\Phi_s(\varphi, \psi)f := \varphi(f)^s \psi(f) = (\varphi(f) \circ \varphi(f) \circ \cdots \circ \varphi(f))(\psi(f)).$$

This is a homogeneous covariant of degree $\deg \Phi_s = s \deg \varphi + \deg \psi$.

Let $\mathfrak{sl}_2 := \mathrm{Lie} \mathrm{SL}_2$ be the Lie algebra of SL_2 which acts on a representation V of SL_2 by the adjoint representation $\mathrm{ad}: \mathfrak{sl}_2 \rightarrow \mathrm{End}(V)$. As an SL_2 -module we have $\mathfrak{sl}_2 \xrightarrow{\sim} V_2$, and $\mathfrak{sl}_2[2] = \mathbb{k} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Lemma 6.4. *Let V_d denote the binary forms of degree d , considered as a representation of SL_2 .*

- (1) If d is odd, then there is a quadratic covariant $\varphi_0: V_d \rightarrow \mathfrak{sl}_2$ such that $\varphi_0(V_d[1]) = \mathfrak{sl}_2[2] = \mathbb{k} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
- (2) If d is even, then there is a quadratic covariant $\varphi_0: V_d \rightarrow \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$ such that $\varphi_0(V_d[2]) = \mathfrak{sl}_2[2] \otimes \mathfrak{sl}_2[2]$.
- (3) If $d \equiv 0 \pmod{4}$, then there is a quadratic covariant $\psi: V_d \rightarrow V_d$ such that $\psi(V_d[2]) = V_d[4]$.
- (4) If $d \equiv 2 \pmod{4}$ and $d \geq 10$, then there is a homogeneous covariant $\psi: V_d \rightarrow V_d$ of degree 4 such that $\psi(V_d[2]) = V_d[8]$, and there is no quadratic covariant.

For the proof let us recall the CLEBSCH-GORDAN-decomposition of the tensor product $V_d \otimes V_e$ as an SL_2 -module where we assume that $d \geq e$:

$$V_d \otimes V_e \simeq \bigoplus_{r=0}^e V_{d+e-2r}.$$

The projection $\tau_r: V_d \otimes V_e \rightarrow V_{d+e-2r}$ is classically called the r th *transvection*, and it is given by the following formula:

$$(T1) \quad f \otimes h \mapsto (f, h)_r := \sum_{i=0}^r (-1)^i \binom{r}{i} \frac{\partial^r f}{\partial x^{r-i} \partial y^i} \frac{\partial^r h}{\partial x^i \partial y^{r-i}}.$$

The second symmetric power $S^2(V_d)$ has the decomposition

$$S^2(V_d) \xrightarrow{\sim} V_{2d} \oplus V_{2d-4} \oplus V_{2d-8} \oplus \cdots,$$

and thus the quadratic covariants $f \mapsto (f, f)_r$ are non-zero only for even r , and are given by

$$(T2) \quad (f, f)_r := \sum_{i=0}^r (-1)^i \binom{r}{i} \frac{\partial^r f}{\partial x^{r-i} \partial y^i} \frac{\partial^r f}{\partial x^i \partial y^{r-i}} \in V_{2d-2r}.$$

Lemma 6.5. *For $d = 2m$ and an even $r < d$, the transvection*

$$(x^{m-1}y^{m+1}, x^{m-1}y^{m+1})_r \in V_{2d-2r}$$

is a nonzero multiple of $x^{d-r-2}y^{d-r+2}$.

Proof. For $r < d = 2m$ we have $(x^{m-1}y^{m+1}, x^{m-1}y^{m+1})_r = c_{m,r} x^{d-r-2} y^{d-r+2}$,

$$(*) \quad c_{m,r} = \sum_{i=r-m+1}^{m-1} (-1)^i \binom{r}{i} (m-r+i)_{r-i} (m-i+2)_i (m-i)_i (m-r+i+2)_{r-i}$$

where $(x)_n := x(x+1)(x+2) \cdots (x+n-1)$. Using Mathematica [Res16], we find

$$c_{m,r} = (-1)^{m+r-1} 4^{m-1} \frac{\Gamma(m)\Gamma(m+2)\Gamma(2m-\frac{r}{2}+1)\Gamma(\frac{r+1}{2})}{\Gamma(2m-r-1)\Gamma(2m-r+1)\Gamma(m-\frac{r}{2}+2)\Gamma(-m+\frac{r+3}{2})}.$$

Using some standard functional equations for Γ we get for $m, s \in \mathbb{N}$, $m \geq 1$ and $0 \leq r = 2s < d = 2m$:

$$(**) \quad \begin{aligned} c_{m,2s} &= (-1)^s \frac{(2s)!(m-1)!(m+1)!(2m-s)!}{s!(m-s-1)!(m-s+1)!(2m-2s)!} = \\ &= (-1)^s (2s)!(s!)^2 \binom{m-1}{s} \binom{m+1}{s} \binom{2m-s}{s} \neq 0 \end{aligned}$$

□

Remark 6.6. With a similar computation one shows that for $d = 2m + 1$ and an even $r < d$ the transvection $(x^m y^{m+1}, x^m y^{m+1})_r$ is a nonzero multiple of $x^{2m-r} y^{2m-r+2}$. But this will not be used in the following.

Proof of Lemma 6.4. (a) If $d = 2m + 1$, then the $2m$ -th transvection is a covariant $\tau_{2m}: V_d \rightarrow V_2 \simeq \mathfrak{sl}_2$, and $\tau_{2m}(x^m y^{m+1})$ is a non-zero multiple of y^2 . In fact, for $r = 2m$, the sum (T2) has a single term, namely for $i = m$. This proves (1).

(b) Now assume that d is even, $d = 2m$. Then the transvection $\tau_{2m-2}: V_d \rightarrow V_4$ has the property that $\tau_{2m-2}(x^{m-1} y^{m+1})$ is a non-zero multiple of y^4 . In fact, the sum (T2) has a single term, namely for $i = m - 1$. Since $\mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \simeq V_0 \oplus V_2 \oplus V_4$ and $(\mathfrak{sl}_2 \otimes \mathfrak{sl}_2)[4] = \mathfrak{sl}_2[2] \otimes \mathfrak{sl}_2[2]$, we thus get a covariant $\varphi_0: V_d \rightarrow \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$ as claimed in (2).

(c) If m is even, then, again by Lemma 6.5, $\psi := \tau_m: V_d \rightarrow V_d$ is a quadratic covariant such that $\psi(V_d[2]) = V_d[4]$, proving (3). Here we only use that $c_{m,m} \neq 0$.

(d) Finally, if m is odd, $m = 2k + 1$, there is no quadratic covariant, because V_d does not appear in the decomposition of $S^2(V_d)$. But, for $m \geq 5$, there is a homogeneous covariant ψ of degree 4 with the required property. For even k we take

$$\psi: V_d \rightarrow V_d, f \mapsto ((f, f)_{3k}, (f, f)_{3k+2})_1,$$

and for odd k

$$\psi: V_d \rightarrow V_d, f \mapsto ((f, f)_{3k-1}, (f, f)_{3k+3})_1.$$

By Lemma 6.5, $(x^{m-1} y^{m+1}, x^{m-1} y^{m+1})_r$ is a nonzero multiple of $x^{2m-r-1} y^{2m-r+1}$ for even r such that $0 \leq r \leq m - 1$. It remains to see that the transvections $(x^k y^{k+4}, x^{k-2} y^{k+2})_1$ for even k and $(x^{k+1} y^{k+5}, x^{k-3} y^{k+1})_1$ for odd k are nonzero. This follows from the transvection formula (T1) above which gives

$$\begin{aligned} (x^k y^{k+4}, x^{k-2} y^{k+2})_1 &= 8 \cdot x^{2k-3} y^{2k+5} = 8 \cdot x^{m-4} y^{m+4}, \\ (x^{k+1} y^{k+5}, x^{k-3} y^{k+1})_1 &= 16 \cdot x^{2k-3} y^{2k+5} = 16 \cdot x^{m-4} y^{m+4}. \end{aligned}$$

This proves (4). \square

Remark 6.7. Each summand in the expression (*) for $c_{m,r}$ can be rewritten as a certain multiple of a product of three binomial coefficients, and one obtains

$$c_{m,r} = \frac{(m-1)!(m+1)!r!}{(2m-r)!} \sum_{i=r-m+1}^{m-1} (-1)^i \binom{m-1}{r-i} \binom{m+1}{i} \binom{2m-r}{m-i-1}.$$

In case of $r = m$ we find

$$(***) \quad c_{m,m} = \frac{(m+1)!^2}{m^2} \sum_{i=1}^{m-1} (-1)^i \binom{m}{i-1} \binom{m}{i} \binom{m}{i+1}.$$

KONVALINKA has shown (see [Kon08, formula 1.1]) that this alternating sum is nonzero for even m . This implies that we have a rigorous proof for the first three statements of Lemma 6.4, because only in part (d) of the proof we used the non-vanishing of a general $c_{m,r}$.

6.4. Symmetric subspaces of the nullforms. We will now determine the minimal SL_2 -symmetric subspaces of the nullforms $\mathcal{N}(V_d)$ and calculate the first integrals.

Theorem 6.8. *Let $d = 2m + 1$ be odd, $d \geq 3$.*

- (1) $d(\mathcal{N}(V_d)) = m$.
- (2) V_d^+ is a minimal SL_2 -symmetric subspace of $\mathcal{N}(V_d)$ of dimension m .
- (3) If $M \subseteq \mathcal{N}(V_d)$ is a minimal SL_2 -symmetric subspace of dimension m , then $M = gV_d^+$ for some $g \in \mathrm{SL}_2$.
- (4) $\mathcal{N}(V_d) // \mathrm{End}_{\mathrm{SL}_2}(V_d) \simeq \mathrm{SL}_2 / B \simeq \mathbb{P}^1$.
- (5) $\mathcal{F}_{\mathrm{SL}_2}(\mathcal{N}(V_d)) \simeq \mathbb{k}(\mathrm{SL}_2 / B)$, in particular $\mathcal{F}_{\mathrm{SL}_2}(\mathcal{N}(V_d))^{\mathrm{SL}_2} = \mathbb{k}$.

Proof. (a) Consider the covariants $\Phi_s(\varphi, \mathrm{id}): V_d \rightarrow V_d$ defined above where φ is the composition

$$\varphi: V_d \xrightarrow{\varphi_0} \mathfrak{sl}_2 \xrightarrow{\mathrm{ad}} \mathrm{End}(V_d)$$

and $\varphi_0: V_d \rightarrow \mathfrak{sl}_2$ is from Lemma 6.4(1). By construction, we get

$$\Phi_s(V_d[1]) = \mathrm{ad} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^s V_d[1] = V_d[2s + 1].$$

This shows that $\mathrm{End}_{\mathrm{SL}_2}(V_d)(V_d[1]) = V_d^+$, hence (1) and (2).

(b) Let $M = M(f)$ be of dimension m . There is a $g \in \mathrm{SL}_2$ such that $gf \in V_d^+$, hence $gM(f) = M(gf) \subseteq V_d^+$. Since $\dim M(f) = m$ we get $gM(f) = V_d^+$. This gives (3) and shows that SL_2 acts transitively on the subspaces $M(f) \subseteq \mathcal{N}(V_d)$ of dimension m and thus on the image of $\pi: \mathcal{N}(V_d) \rightarrow \mathrm{Gr}_m(V_d)$. Since the normalizer of V_d^+ is B , we finally get (4) and (5). \square

Theorem 6.9. *Let $d = 2m$ and m even.*

- (1) $d(\mathcal{N}(V_d)) = m$.
- (2) V_d^+ is a minimal SL_2 -symmetric subspace of $\mathcal{N}(V_d)$ of dimension m .
- (3) If $M \subseteq \mathcal{N}(V_d)$ is a minimal SL_2 -symmetric subspace of dimension m , then $M = gV_d^+$ for some $g \in \mathrm{SL}_2$.
- (4) $\mathcal{N}(V_d) // \mathrm{End}_{\mathrm{SL}_2}(V_d) \simeq \mathrm{SL}_2 / B \simeq \mathbb{P}^1$.
- (5) $\mathcal{F}_{\mathrm{SL}_2}(\mathcal{N}(V_d)) \simeq \mathbb{k}(\mathrm{SL}_2 / B)$, in particular $\mathcal{F}_{\mathrm{SL}_2}(\mathcal{N}(V_d))^{\mathrm{SL}_2} = \mathbb{k}$.

Proof. Define the following covariant

$$\varphi: V_d \xrightarrow{\varphi_0} \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \xrightarrow{\alpha} \mathrm{End}(V_d)$$

where φ_0 is from Lemma 6.4(2), and α is the linear SL_2 -equivariant map $A \otimes B \mapsto \mathrm{ad} A \circ \mathrm{ad} B$. Then the covariants $\Phi_s(\varphi, \mathrm{id}): V_d \rightarrow V_d$ satisfy $\Phi_s(V_d[2]) = (\mathrm{ad} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})^{2s} V_d[2] = V_d[4s + 2]$, and for the covariants $\Phi_s(\varphi, \psi)$ where ψ is from Lemma 6.4(3) we get $\Phi_s(V_d[2]) = \mathrm{ad} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^{2s} V_d[4] = V_d[4s + 4]$. As a consequence, we get $\mathrm{End}_{\mathrm{SL}_2}(V_d)(V_d[2]) = V_d^+$, hence (1) and (2). The remaining claims follow as in the proof of Theorem 6.8. \square

If $d = 2m$ and m odd we define $V_d^{++} := V_d[2] \oplus V_d[6] \oplus V_d[8] \oplus \cdots$.

Theorem 6.10. *Let $d = 2m$ and m odd, $m \geq 3$.*

- (1) $d(\mathcal{N}(V_d)) = m - 1$.
- (2) V_d^{++} is a minimal SL_2 -symmetric subspace of $\mathcal{N}(V_d)$ of dimension $m - 1$.
- (3) If $M \subseteq \mathcal{N}(V_d)$ is a minimal SL_2 -symmetric subspace of dimension $m - 1$, then $M = gV_d^{++}$ for some $g \in \mathrm{SL}_2$.
- (4) $\mathcal{N}(V_d) // \mathrm{End}_{\mathrm{SL}_2}(V_d) \simeq \mathrm{SL}_2 / T$.

(5) $\mathcal{F}_{\mathrm{SL}_2}(\mathcal{N}(V_d)) \simeq \mathbb{k}(\mathrm{SL}_2/T)$, in particular $\mathcal{F}_{\mathrm{SL}_2}(\mathcal{N}(V_d))^{\mathrm{SL}_2} = \mathbb{k}$.

Proof. (a) We first remark that there is no quadratic covariant $\varphi: V_d \rightarrow V_d$, and so V_d^{++} is stable under $\mathcal{E} := \mathrm{End}_{\mathrm{SL}_2}(V_d)$. Now we use the covariants $\Phi_s(\varphi, \mathrm{id})$, as in the proof of the previous theorem, to show that $\mathcal{E}(V_d[2]) \supset V_d[4s+2]$. Moreover, the covariants $\Phi_s(\varphi, \psi)$ with ψ from Lemma 6.4(4) imply that $\mathcal{E}(V_d[2]) \supset V_d[4s+8]$. It follows that $\mathcal{E}(V_d[2]) = V_d^{++}$, hence (1) and (2).

(b) Using again that there are no quadratic covariants, we see that $\mathcal{E}(V_d[4]) \subseteq V_d[4] \oplus V_d[8] \oplus V_d[10] \oplus \dots$, hence $\dim \mathcal{E}(V_d[4]) \leq m-2$. Therefore, V_d^{++} is the only minimal SL_2 -symmetric subspace of V_d^+ of dimension $m-1$. Now the remaining claims follow as before, using that the normalizer of V_d^{++} is T . \square

Remark 6.11. Part (4) of Lemma 6.4 was only used in the proof of the last theorem. Hence we have a rigorous proof of Theorem 6.8 and Theorem 6.9, independent of the symbolic calculations done with Mathematica (see Remark 6.7).

Example 6.12. The minimal orbit $O_0 \subseteq V_d$ is the orbit of y^d . Denote by O_1 the orbit of xy^{d-1} . Then $X := \overline{O_1} = O_1 \cup O_0 \cup \{0\}$. We claim that X is SL_2 -symmetric and that $\mathrm{End}_{\mathrm{SL}_2}(X) = \mathbb{k} \cdot \mathrm{id}$ in case $d \geq 5$. In fact, the image of $xy^{d-1} \in V_d^+$ under $\varphi \in \mathrm{End}_{\mathrm{SL}_2}(X)$ is again a weight vector of positive weight, hence a multiple of some $x^\ell y^{d-\ell}$ where $\ell < d-\ell$. Since the stabilizer of $x^\ell y^{d-\ell}$ in SL_2 is cyclic of order $d-2\ell$ for $\ell < d-\ell$, we see that $\varphi(xy^{d-1})$ is a multiple of xy^{d-1} if $d > 4$. (For $d = 4$, X is the nullcone $\mathcal{N}(V_4)$, and the quadratic covariant φ sends O_1 onto O_0 , see Theorem 6.9.) This implies that $\varphi|_{O_1} = \lambda \cdot \mathrm{id}$ for some $\lambda \in \mathbb{k}$, hence $\varphi|_X = \lambda \cdot \mathrm{id}$. As a consequence, $X' = X \setminus \{0\}$, and $X//\mathcal{E} = \mathbb{P}(X) \subseteq \mathbb{P}(V_d)$.

Example 6.13. Let $d = 2m$ be even and consider V_d^+ as a B -module. It is not difficult to see that there is always a B -covariant φ of degree 2. E.g. for $d = 6$ it is given by

$$\varphi(a_1 \cdot x^2 y^4 + a_2 \cdot xy^5 + a_3 \cdot y^6) = 2a_1^2 \cdot xy^5 + a_1 a_2 \cdot y^6.$$

On the other hand, for $d = 2m \geq 6$ and m odd there is no SL_2 -covariant of V_d of degree 2 (Lemma 6.4(4)). Since $\mathrm{End}_{\mathrm{SL}_2}(V) = \mathrm{End}_B(V)$ for every SL_2 -module V , we see that the restriction map $\mathrm{End}_B(V_d) \rightarrow \mathrm{End}_B(V_d^+)$ for $d \equiv 2 \pmod{4}$ and $d \geq 6$ is not surjective.

APPENDIX: IND-VARIETIES AND IND-SEMIGROUPS

An introduction to ind-varieties and ind-groups can be found in KUMAR's book [Kum02, Chapter IV].

7.1. Basic definitions. The following is borrowed from [FK16].

Definition 7.14. An *ind-variety* \mathcal{V} is a set together with an ascending filtration $\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \dots \subseteq \mathcal{V}$ such that the following holds:

- (1) $\mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}_k$;
- (2) Each \mathcal{V}_k has the structure of an algebraic variety;
- (3) For all $k \in \mathbb{N}$ the inclusion $\mathcal{V}_k \hookrightarrow \mathcal{V}_{k+1}$ is closed immersion of algebraic varieties.

A *morphism* between ind-varieties \mathcal{V} and \mathcal{W} is a map $\varphi: \mathcal{V} \rightarrow \mathcal{W}$ such that for any k there is an m such that $\varphi(\mathcal{V}_k) \subseteq \mathcal{W}_m$ and that the induced map $\mathcal{V}_k \rightarrow \mathcal{W}_m$ is a morphism of varieties. *Isomorphisms* of ind-varieties are defined in the obvious way.

Two filtrations $\mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}_k$ and $\mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}'_k$ are called *equivalent* if for any k there is an m such that $\mathcal{V}_k \subseteq \mathcal{V}'_m$ is a closed subvariety as well as $\mathcal{V}'_k \subseteq \mathcal{V}_m$. Equivalently, the identity map $\text{id}: \mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}_k \rightarrow \mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}'_k$ is an isomorphism of ind-varieties.

Definition 7.15. The *Zariski topology* of an ind-variety $\mathcal{V} = \bigcup_k \mathcal{V}_k$ is defined by declaring a subset $U \subseteq \mathcal{V}$ to be open if the intersections $U \cap \mathcal{V}_k$ are Zariski-open in \mathcal{V}_k for all k . It is obvious that $A \subseteq \mathcal{V}$ is closed if and only if $A \cap \mathcal{V}_k$ is Zariski-closed in \mathcal{V}_k for all k . It follows that a locally closed subset $\mathcal{W} \subseteq \mathcal{V}$ has a natural structure of an ind-variety, given by the filtration $\mathcal{W}_k := \mathcal{W} \cap \mathcal{V}_k$ which are locally closed subvarieties of \mathcal{V}_k . These subsets are called *ind-subvarieties*.

A morphism $\varphi: \mathcal{V} \rightarrow \mathcal{W}$ is called an *immersion* if the image $\varphi(\mathcal{V}) \subseteq \mathcal{W}$ is locally closed and φ induces an isomorphism $\mathcal{V} \xrightarrow{\sim} \varphi(\mathcal{V})$ of ind-varieties. An immersion φ is called a *closed* (resp. *open*) *immersion* if $\varphi(\mathcal{V}) \subseteq \mathcal{W}$ is closed (resp. open).

Definition 7.16.

- (1) An ind-variety \mathcal{V} is called *affine* if it admits a filtration such that all \mathcal{V}_k are affine. It follows that any filtration of \mathcal{V} has this property.
- (2) The *algebra of regular functions* on $\mathcal{V} = \bigcup_k \mathcal{V}_k$ is defined as

$$\mathcal{O}(\mathcal{V}) := \text{Mor}(\mathcal{V}, \mathbb{A}^1) = \varprojlim \mathcal{O}(\mathcal{V}_k)$$

It will always be regarded as a topological algebra with the obvious topology as an inverse limit of finitely generated algebras. For any morphism $\varphi: \mathcal{V} \rightarrow \mathcal{W}$ the induced homomorphism $\varphi^*: \mathcal{O}(\mathcal{W}) \rightarrow \mathcal{O}(\mathcal{V})$ is continuous. Moreover, an affine ind-variety \mathcal{V} is uniquely determined by the topological algebra $\mathcal{O}(\mathcal{V})$.

- (3) The *Zariski tangent space* of an ind-variety $\mathcal{V} = \bigcup_k \mathcal{V}_k$ is defined in the obvious way:

$$T_v \mathcal{V} := \varinjlim T_v \mathcal{V}_k.$$

If \mathcal{V} is affine, then a tangent vector $A \in T_v \mathcal{V}$ is the same as a continuous derivation $A: \mathcal{O}(\mathcal{V}) \rightarrow \mathbb{k}$ in v . It is clear that a morphism $\varphi: \mathcal{V} \rightarrow \mathcal{W}$ between two ind-varieties induces a linear map $d\varphi_v: T_v \mathcal{V} \rightarrow T_{\varphi(v)} \mathcal{W}$, the *differential of φ in v* .

- (4) The *product of two ind-varieties* $\mathcal{V} = \bigcup_k \mathcal{V}_k$ and $\mathcal{W} = \bigcup_j \mathcal{W}_j$ is the ind-variety defined as $\mathcal{V} \times \mathcal{W} := \bigcup_k \mathcal{V}_k \times \mathcal{W}_k$. It has the usual universal properties.
- (5) An ind-variety \mathcal{V} is *curve-connected* if for every pair $v, w \in \mathcal{V}$ there is an irreducible algebraic curve C and a morphism $\gamma: C \rightarrow \mathcal{V}$ such that $v, w \in \gamma(C)$. One can show that this is equivalent to the existence of a filtration $\mathcal{V} = \bigcup_k \mathcal{V}_k$ such that all \mathcal{V}_k are irreducible (see [FK16]).

Since products exist in the category of ind-varieties we can define ind-groups and ind-semigroups.

Definition 7.17. An *ind-group* \mathcal{G} is an ind-variety with a group structure such that multiplication $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ and inverse $\mathcal{G} \rightarrow \mathcal{G}$ are morphisms. An *ind-semigroup* \mathcal{S} is defined in a similar way.

An *action of an ind-group \mathcal{G} on a variety X* is a homomorphism $\mathcal{G} \rightarrow \text{Aut}(X)$ such that the induced map $\mathcal{G} \times X \rightarrow X$ is a morphism of ind-varieties. If X is an affine variety, it is shown in [FK16] that $\text{End}(X)$ is an affine ind-semigroup and $\text{Aut}(X)$ is an affine ind-group which is locally closed in $\text{End}(X)$. It follows that an action of an ind-group \mathcal{G} on X is the same as a homomorphism of ind-groups $\mathcal{G} \rightarrow \text{Aut}(X)$.

All this carries over to actions of ind-semigroups \mathcal{S} .

7.2. Vector fields and Lie algebras. A *vector field* δ on an affine variety X is a collection $\delta = (\delta(x))_{x \in X}$ of tangent vectors $\delta(x) \in T_x X$ such that, for all $f \in \mathcal{O}(X)$, we have $\delta f \in \mathcal{O}(X)$ where $(\delta f)(x) := \delta(x)f$. It follows that the vector fields $\text{Vec}(X)$ can be identified with the derivations of $\mathcal{O}(X)$ which we denote by $\text{Der}(\mathcal{O}(X))$.

The same definition can be used for an affine ind-variety \mathcal{V} , and one gets an identification of $\text{Vec}(\mathcal{V})$ with the *continuous* derivations $\text{Der}^c(\mathcal{O}(\mathcal{V}))$. For an affine ind-group \mathcal{G} one shows that the tangent space $T_e \mathcal{G}$ has a natural structure of a Lie algebra. It will be denoted by $\text{Lie } \mathcal{G}$.

If \mathcal{G} acts on the variety X and $x \in X$ we denote by $\mu_x: \mathcal{G} \rightarrow X$ the orbit map $g \mapsto gx$.

Proposition 7.18. *Assume that an affine ind-group \mathcal{G} acts on an affine variety X . For $A \in \text{Lie } \mathcal{G}$ and $x \in X$ define the tangent vector $\xi_A(x) \in T_x X$ to be the image of A under $d\mu_x: \text{Lie } \mathcal{G} \rightarrow T_x X$. Then ξ_A is a vector field on X . The resulting linear map $\Xi: \text{Lie } \mathcal{G} \rightarrow \text{Vec}(X)$, $A \mapsto \xi_A$, is a anti-homomorphism of Lie algebras.*

Outline of Proof. The action $\varphi: \mathcal{G} \times X \rightarrow X$ defines a homomorphism $\varphi^*: \mathcal{O}(X) \rightarrow \mathcal{O}(\mathcal{G}) \otimes \mathcal{O}(X)$. Now consider the following derivation of $\mathcal{O}(X)$:

$$\delta: \mathcal{O}(X) \xrightarrow{\varphi^*} \mathcal{O}(\mathcal{G}) \otimes \mathcal{O}(X) \xrightarrow{A \otimes \text{id}} \mathcal{O}(X).$$

An easy calculation shows that $(\delta f)(x) = A\mu_x^*(f) = d\mu_x(A)f$, hence $\delta = \xi_A$. \square

It is easy to see that this generalizes to the action of an affine ind-semigroup \mathcal{E} on an affine variety X , $\mu: \mathcal{E} \rightarrow \text{End}(X)$, and defines a linear map $\Xi: T_{\text{id}} \mathcal{E} \rightarrow \text{Vec}(X)$ whose image will be denoted by $\mathcal{D}_{\mathcal{E}}$ and called the *corresponding vector fields*.

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